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Connecting concept lattices with bonds induced by external information

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ABSTRACT

In Formal Concept Analysis (FCA), $\mathcal{L}$-bonds represent relationships between $\mathcal{L}$-formal contexts. Choosing the appropriate bond between $\mathcal{L}$-fuzzy formal contexts is an important challenge for its application in recommendation tasks. Recent work introduced two constructions of bonds, given by direct products of two $\mathcal{L}$-fuzzy formal contexts, and showed their usefulness in a particular application. In this paper, we present further theoretical and experimental results on these constructions; in particular, we provide extended interpretations of both rigorous and benevolent concept-forming operators, introduce new theoretical properties of the proposed bonds to connect two concept lattices given external information, and finally present the experimental study of the upper bounds.

1. Introduction

Formal Concept Analysis (FCA) [16,39] is a method for deriving implicit relationships between objects described by a set of attributes on the one hand and these attributes on the other. FCA can be seen as a conceptual clustering technique as it also provides intensional descriptions for the abstract concepts or data units it produces. Furthermore, it provides powerful theoretical and practical techniques for solving various issues of supervised and unsupervised learning tasks. Recent studies present different perspectives of FCA from the point of view of Artificial Intelligence, as it can provide powerful techniques for knowledge representation, classification tasks, recommendation techniques, or data analysis in general; several researchers have described interesting connections between FCA and methods of machine learning and artificial intelligence [28,29,38], and its potential applications [14,36]. Moreover, the interpretability of feed-forward neural networks learning results can be improved by the techniques of FCA [30], which can be applied for example in interpretable classification of documents based on concept lattices and graph theory [33].

Central to FCA is the notion of a *formal context*, which is a triple (B, A, r), where B is a set of objects, A is a set of attributes, and r is a relation in $B \times A$ indicating which objects have which attributes. Then, a formal concept is defined as a pair of mutually inter-definable subsets of objects and attributes. Finally, the set of all formal concepts of a formal context can be proved to form a complete lattice, called the *concept lattice* of the context.

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Fuzzy generalizations of FCA have been thoroughly studied by several research groups both from its theoretical and practical aspects, see for example the interesting results of Bělohlávek [5,6,9], Ben Yahia and Jaoua [11], Pollandt [37], Burusco and Fuentes-González [12], Medina and Ojeda-Aciego [32], Pócs, Pócsová, Butka [13,34,35]. In addition, we proposed several approaches which extend the FCA from the fuzzy point of view [1,2,23], as well.

The connections between two formal contexts (or two concept lattices) were thoroughly investigated shortly after the first results of FCA were published by Rudolph Wille [39]. Then, Ganter and Wille [16] explored the maps between concept lattices to compare the contexts and to provide the conceptual measurability. The notion of *bond* between formal contexts was defined in connection with subdirect products of concept lattices. Regarding the question of bonds between $\mathcal{L}$-fuzzy formal contexts, a systematic investigation was provided by Konečný et al. [19–22], and also by our papers [24–27].

The contributions in this paper follow the line of other researchers in the area of isotone concept-forming operators [18,17,4] and the composition of fuzzy relations [8].

Motivation

Given two formal contexts with related data, our main objective is to find a possible connection between the sets of objects of each context.

One possible approach could be based on the assumption that both contexts have similar attributes and this similarity is given by an external relation between attribute sets. For example, imagine two formal contexts of people and their hobbies. The aim here would be to connect these two groups of people by looking at the similarities in their hobbies.

For a second approach, it is also possible to consider the two formal contexts with different types of objects; for example, the first formal context could be a relationship between users of a music streaming application and the music albums they have enjoyed in the past; and the second formal context could be the actually released music albums and their relationship to older albums or information about their music genre or similar relationships. The aim here would be to offer users the new albums taking into account their musical genre and/or their relationship to older albums according to the users' musical preferences.

A third example could be based on two formal contexts, where the first context includes the results of secondary school students in their subjects, and the second context includes the study programmes of universities and the scientific fields of these study programmes. The aim is to find the connection between students and study programmes based on the similarity between subjects and scientific fields, and further on the interests of students and scientific fields of study programmes. This third example will be used as a running example throughout the paper.

The main contribution of our paper is to extend previous results in the area of selecting the most appropriate bonds given by direct products of two $\mathcal{L}$-fuzzy formal contexts [24]. Specifically, we present the additional theoretical and experimental results and interpretations for the fuzzy concept-forming operators and the selection of appropriate bonds, including studying the quality of the solution.

The rest of the paper is structured as follows: Section 2 includes the preliminaries including the definitions of concept-forming operators in a fuzzy setting. We extend the known results by additional explanations and interpretations of rigorous and benevolent concept-forming operators. In Section 3, we provide the extended interpretations of rigorous and benevolent methods for selecting the bonds which connect the concept lattices. New theoretical results of both proposed approaches, the rigorous and the benevolent, connecting two formal contexts (or concept lattices) given external information are presented in Section 4. Finally, the experimental study of the upper bounds is included in Section 5.

2. Preliminaries

We firstly remind the basic notions of Formal Concept Analysis [16] and its $\mathcal{L}$-fuzzy generalisation [5,6,9].

Our underlying algebra of degrees which will describe the relationship between elements is that of a *complete residuated lattice* $\mathcal{L} = (L, 1, 0, \wedge, \vee, \otimes, \to)$. This means that $(L, 1, 0, \wedge, \vee)$ is a complete lattice with top element 1 and bottom element 0. Values from such a structure allow us to express degrees, the easiest example being to consider L as the unit interval; the possibility of considering any lattice structure enables us to consider incomparable values for interpreting more complex relationships than using linearly ordered structures.

The operations $\otimes\colon L \times L \longrightarrow L$ and $\to\colon L \times L \longrightarrow L$ provide logical content to the lattice. It is required that $(L, \otimes, 1)$ forms a commutative monoid, i.e. $\otimes$ is commutative binary operation with 1 as neutral element, and $\to$ is the residual implication of $\otimes$, that is, for any three values $(k, l, m) \in L^3$ it holds that $k \le l \to m \iff k \otimes l \le m \iff l \le k \to m$. Hence, $\otimes$ and $\to$ act like conjunction and implication on such a generalised structure.

The following properties of complete residuated lattices will be used later in the proofs of the theoretical results: given an index set $i \in I$, for all $l, m, l_i, m_i \in L$, it holds

$$\bigwedge_{i\in I}(l_i \to m) = \bigvee_{i\in I} l_i \to m \qquad\qquad l \to \bigwedge_{i\in I} m_i = \bigwedge_{i\in I} l \to m_i.$$

Given a universe set U and a complete residuated lattice $\mathcal{L}$, we can consider the $\mathcal{L}$-fuzzy sets of U as mappings $U \to L$, and the set of all the $\mathcal{L}$-fuzzy sets of U is denoted as $\mathcal{L}^U$.

An $\mathcal{L}$-context is a triple (B, A, r) formed by two finite sets B and A (the so-called *objects* and *attributes*), and a mapping $r\colon B \times A \longrightarrow L$ which can be interpreted as an $\mathcal{L}$-fuzzy relation between objects and attributes, i.e. this gives us the degree of relationship between any object-attribute pair $(b, a) \in B \times A$.

2.1. Concept-forming operators

Given an $\mathcal{L}$-context (B, A, r), the *concept-forming operators* (or *derivation operators*, or *polars*) can be given in *antitone* or in *isotone* form, which can be also interpreted as so-called *rigorous* and *benevolent* derivation operations, respectively.

2.1.1. Rigorous concept-forming operators

In the rest of the paper, the rigorous concept-forming operators will be denoted by $\uparrow : L^B \longrightarrow L^A$ and $\downarrow : L^A \longrightarrow L^B$. They are defined as follows:

$$\uparrow f(a) = \bigwedge_{b\in B}(f(b) \to r(b,a)) \text{ and } \downarrow g(b) = \bigwedge_{a\in A}(g(a) \to r(b,a))$$

where $f \in L^B$ and $g \in L^A$ are arbitrary $\mathcal{L}$-fuzzy sets of objects or attributes.

Any pair $(f, g) \in L^B \times L^A$ such that $f = \downarrow g$ and $g = \uparrow f$ is called *formal $\mathcal{L}$-fuzzy concept constructed by rigorous concept-forming operators*. The object part f is called *extent*, and the attribute part g is called *intent*. All $\mathcal{L}$-fuzzy concepts can be ordered by $\mathcal{L}$-subset inclusion in the object part, which is equivalent to the dual inclusion order of the attribute part, and forms a complete lattice called *concept lattice*, denoted by $\mathrm{CL}(B, A, r, \uparrow, \downarrow)$.

The pair of operators $(\uparrow, \downarrow)$ defined above forms an *antitone Galois connection between* $(L^B, \le)$ and $(L^A, \le)$ where $\le$ is the $\mathcal{L}$-subset inclusion relation. This means that, for any $f \in L^B$ and any $g \in L^A$, the equivalence $f \le \downarrow g \iff g \le \uparrow f$ holds. It is well-known that compositions $\uparrow \circ \downarrow$ and $\downarrow \circ \uparrow$ form *closure operators* on the respective $\mathcal{L}$-powerset complete lattices.

Interpretation of rigorous concept-forming operators. Consider the set of objects B as a set of students of secondary school, and the set of attributes A as a set of their subjects. Consider also that our underlying residuated lattice is a scale of three values $(0, 0.5, 1)$. The relation r between the sets of students and subjects represents the information about student results for the subjects (value 1 means excellent result, 0.5 corresponds to average results, and 0 means poor results).

Let g be an $\mathcal{L}$-fuzzy set of subjects describing the minimal abilities (requirements) required on students that would like to choose some specific study programme at some university in the future. Then $\downarrow g(b) = \bigwedge_{a\in A}(g(a) \to r(b,a))$ is the value of how the minimal requirements of the study programme are fulfilled by the results of student $b \in B$; hence, a computed value equal to 1 means that all of her/his studying results satisfy the minimal requirements of the study programme: such a student is a good candidate for this study program. If there is a subject a for which the result $r(b,a)$ of student b is less than the requirement $g(a)$, then $g(a) \to r(b,a)$ is less than 1. This will affect the infimum for all attributes. Hence, the final value of $\downarrow g(b)$ will be less than 1, as well. Note that $\downarrow g$ is an $\mathcal{L}$-set of students who meet the minimal requirements for access to the study programme described by g.

Similarly, $\uparrow f$ is an $\mathcal{L}$-set of minimal abilities (requirements) common to all students of $\mathcal{L}$-set f. If each student $b \in B$ satisfies the requirements with $f(b) = 1$, then given a subject a we have that $\uparrow f(a) = \bigwedge_{b\in B}(f(b) \to r(b,a))$ takes into account (in computing the infimum) of the results of all students for subject a. Notice that this is no longer the case should the students membership values to f be lower than 1, and requirements for some subject a are lower than 1 (e.g. say $\uparrow f(a) = 0.5$), because in this case the results of students could be poor as well (it could happen than $r(b,a)$ could be 0 since $0.5 \to_L 0 = 0.5$).

2.1.2. Rigorous conceptual relationship between $\mathcal{L}$-sets

It is easy to extend the relation r defined on object-attribute pairs, $(b,a) \in B \times A$, to pairs of $\mathcal{L}$-fuzzy sets of objects and attributes.

We can say that the pair of $\mathcal{L}$-fuzzy sets $(e,h) \in \mathcal{L}^B \times \mathcal{L}^A$ is in the relation r if and only if there exists a formal $\mathcal{L}$-concept $(f, g) \in \mathrm{CL}(B, A, r, \uparrow, \downarrow)$ such that $e \le f$ and $h \le g$ (note that the extension is a crisp relation).

As a result, we would have $e \le f = \downarrow g \le \downarrow h$, and this means that $e(b) \le \bigwedge_{a\in A}(h(a) \to r(b,a))$ holds for all $b \in B$, which implies $e(b) \otimes h(a) \le r(b,a)$ for all $(b,a) \in B \times A$. Note that the same inequality is obtained if we had used $h \le \uparrow e$ instead.

Regarding $(e,h) \in \mathcal{L}^B \times \mathcal{L}^A$, the so-called covering concepts can be determined by $(\downarrow h, \uparrow\downarrow h)$ or $(\downarrow\uparrow e, \uparrow e)$. These covering concepts will be used later in the section on theoretical results.

2.2. Benevolent concept-forming operators

We present a second type of concept-forming operators $\nearrow : \mathcal{L}^B \longrightarrow \mathcal{L}^A$ and $\swarrow : \mathcal{L}^A \longrightarrow \mathcal{L}^B$. The construction of the benevolent concept-forming operators is not symmetric compared to the rigorous ones.

Consider arbitrary $\mathcal{L}$-fuzzy sets $f \in \mathcal{L}^B$ and $g \in \mathcal{L}^A$, then we can define

$$\nearrow f(a) = \bigvee_{b\in B} f(b) \otimes r(b,a) \text{ and } \swarrow g(b) = \bigwedge_{a\in A} r(b,a) \to g(a).$$

Any pair $(f, g) \in L^B \times L^A$ such that $f = \swarrow g$ and $g = \nearrow f$ is called *formal $\mathcal{L}$-fuzzy concept constructed by benevolent concept-forming operators*. Analogously, the object part f is called *extent*, and the attribute part g is called *intent*.

Due to the asymmetry of the construction, it is also possible to define an alternative version $\nwarrow : \mathcal{L}^B \longrightarrow \mathcal{L}^A$ and $\searrow : \mathcal{L}^A \longrightarrow \mathcal{L}^B$ as follows:

$$\nwarrow f(a) = \bigwedge_{b\in B}(r(b,a) \to f(b)) \text{ and } \searrow g(b) = \bigvee_{a\in A}(g(a) \otimes r(b,a)).$$

Both pairs of operators form *isotone* Galois connections between the corresponding $\mathcal{L}$-fuzzy powersets $(L^B, \leq_B)$ and $(L^A, \leq_A)$, in the sense of the following equivalences:

$$f \leq_B \swarrow g \iff \nearrow f \leq_A g \quad \text{and} \quad \searrow g \leq_B f \iff g \leq_A \nwarrow f.$$

The compositions $\nearrow \circ \swarrow$ and $\searrow \circ \nwarrow$ form closure operators, whereas $\swarrow \circ \nearrow$ and $\nwarrow \circ \searrow$ are interior (or kernel) operators.

There is an easy mnemonic to facilitate the reading of these operators: arrows pointing up (resp. down) map objects to attributes (resp. attributes to objects); the construction is based on $(\bigvee, \otimes)$ when the arrow points east and on $(\bigwedge, \to)$ when it points west.

Interpretation of benevolent concept-forming operators. Continuing with our running example, consider B as a set of students and A as a set of secondary school subjects. Consider an $\mathcal{L}$-fuzzy set of topics g that are important for a selected educational competition (i.e., students requirements). The goal is to find a group of people who satisfy outcomes in at least one of the topics in the set g (in a fuzzy sense). In other words, we are looking for students whose results in all subjects have non-empty intersection with the set g, i.e., $\searrow g(b) = \bigvee_{a \in A} g(a) \otimes r(b, a)$.

If the chosen subject result of the chosen student is high (i.e. $r(b, a) = 1$ for subject a and for student b) and if the subject requirement for the educational competition is high (i.e. subject a belongs to g with degree $g(a) = 1$), then the student fulfils the requirements to a high degree (i.e. $g(a) \otimes r(b, a) = 1$); this means that we get $\searrow g(b) = 1$ with the supremum operator. If there is no subject from the intersection of $r(b, -)$ and g, then the value of $\searrow g(b)$ will be less than 1.

The operator $\nwarrow$ of the isotone Galois connection aims to find a subject a such that all students with satisfying results $r(-, a)$ are contained in f. The requirements g can be interpreted as the characteristics that are common to all students in a group (i.e. the strengths of a group). In particular, if the competitive demand for a subject a is high for all students (i.e. $\nwarrow f(a) = 1$ for all $b \in B$), and if the satisfaction with the competitive demand for a subject a is high for all students (i.e. $f(b) = 1$ for all $b \in B$), then the results of all students for that subject a are high (i.e. $r(b, a) = 1$ for all $b \in B$).

The second operator $\searrow$ can generate an $\mathcal{L}$-fuzzy set of students $\searrow g$ (set of students such that at least one of them has satisfying results in at least one subject from g) by an $\mathcal{L}$-fuzzy set of requirements g. If student b has high scores $r(b, a) = 1$ in some subject a and if the requirements for that subject are high (i.e. $g(a) = 1$), then the student fulfils the requirements to a high degree (i.e. $\searrow g(b) = 1$). Concerning the closure of operators $\nwarrow\searrow g$, we ask for the strengths of a set of students $\searrow g$. Of course, the L fuzzy set $\nwarrow\searrow g$ can be a superset of g, which means that the strengths of a set of students can be a superset of the requirements of students.

2.2.1. Benevolent conceptual relationship of $\mathcal{L}$-sets

We can define conceptual benevolent relationship of two $\mathcal{L}$-sets from $\mathcal{L}^B \times \mathcal{L}^A$ in relation $r \in \mathcal{L}^{B \times A}$ as follows: Two sets $(e, h) \in \mathcal{L}^B \times \mathcal{L}^A$ will be called benevolent in relation r if there exist a $\mathcal{L}$-fuzzy concept $(f, g) \in \mathrm{CL}(B, A, r, \nwarrow, \searrow)$ such that $f \leq e$ and $h \leq g$. In particular, we have $\searrow h \leq \searrow g = f \leq e$ and, therefore, for any $(b, a) \in B \times A$ we have

$$r(b, a) \leq h(a) \to e(b).$$

Regarding $(e, h) \in \mathcal{L}^B \times \mathcal{L}^A$, the so-called covering concepts can be determined by $(\searrow h, \nwarrow\searrow h)$ and $(\searrow\nwarrow e, \nwarrow e)$. This type of concepts will be used later in the section of novel theoretical results. The properties of the $\swarrow$ and $\nearrow$ operators can be considered analogously.

2.3. Bonds

Ganter and Wille [16] explored the maps between concept lattices to compare contexts and to provide conceptual measurability. The notion of bond between formal contexts was defined in connection with subdirect products of concept lattices. The several fuzzifications of bonds were proposed by Konečný and by authors of this paper [25,26,19,27,20–22]. We recall below the basic notion of a bond in its weak form. We denote the sets of all extents (and intents) of $\mathcal{L}$-fuzzy formal context $\mathcal{C} = (B, A, r)$ by $\mathrm{Ext}(\mathcal{C})$ (and $\mathrm{Int}(\mathcal{C})$, respectively).

Definition 1. Consider two $\mathcal{L}$-fuzzy formal contexts $\mathcal{C}_1 = (B_1, A_1, r_1)$ and $\mathcal{C}_2 = (B_2, A_2, r_2)$. An $\mathcal{L}$-fuzzy formal context $\mathcal{D} = (B_1, B_2, r)$ is said to be a *bond* between $\mathcal{L}$-fuzzy formal contexts $\mathcal{C}_1$ and $\mathcal{C}_2$ if it holds that $\mathrm{Ext}(\mathcal{D}) \subseteq \mathrm{Ext}(\mathcal{C}_1)$ and $\mathrm{Int}(\mathcal{D}) \subseteq \mathrm{Ext}(\mathcal{C}_2)$.

By Definition 1, a bond between two $\mathcal{L}$-fuzzy formal contexts connects the extents of their $\mathcal{L}$-fuzzy formal concepts. We illustrate the scheme of the bond structures in Fig. 1. Note that alternative forms of bonds are also defined in the literature. In our paper, we will use the bonds based on the extents of the input $\mathcal{L}$-fuzzy formal contexts.

3. Benevolent and rigorous bonds between formal contexts

It is well-known that any extent and intent of direct product is a bond between the input formal contexts. In previous papers [21,25,26], we proved that the structure of complete lattice of all bonds between two formal contexts is isomorphic to all Galois connections between concept lattices of the corresponding input formal contexts. Moreover, we proved that the category of formal contexts and bonds and the category of concept lattices and Galois connections are equivalent [27]; this result is based on the fact that every extent or intent of a dual bond (defined only on object sets of input contexts) is an extent of corresponding formal context.

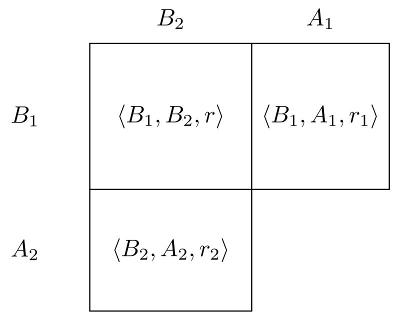

Fig. 1. Scheme of bond structures.

In [24], we described a method of selecting a bond with an input value, i.e. the external information dealing with connecting the attribute sets of input contexts. An output value comes from computing the formal concept over a direct product of two input formal contexts. Specifically, it was proposed the connection of two $\mathcal{L}$-fuzzy formal contexts $C_1 = (B_1, A_1, r_1)$ and $C_2 = (B_2, A_2, r_2)$ (or their concept lattices) in terms of the external information described by a third $\mathcal{L}$-fuzzy formal context (A_1, A_2, p). These constructions can be seen as selecting the most appropriate bond between C_1 and C_2 induced by the external information p.

Now, we will recall the definitions given in [24] of the benevolent and rigorous bonds that connect the concept lattices via two direct products of C_1 and C_2. Moreover, we provide here the extended interpretation of rigorous and benevolent bonds.

3.1. Benevolent method

Given two contexts $C_1 = (B_1, A_1, r_1)$ and $C_2 = (B_2, A_2, r_2)$, we define a new context $C_1 \otimes C_2 = (A_1 \times A_2, B_1 \times B_2, r_\otimes)$ where

$$r_\otimes((a_1, a_2), (b_1, b_2)) = r_1(b_1, a_1) \otimes r_2(b_2, a_2).$$

Now, a benevolent bond between C_1 and C_2 corresponding to external information $p \in \mathcal{L}^{A_1 \times A_2}$ can be obtained as the result of $\nearrow p$ computed on $C_1 \otimes C_2$, that is,

$$\nearrow p(b_1, b_2) = \bigvee_{(a_1,a_2)\in A_1\times A_2} p(a_1, a_2) \otimes r_1(b_1, a_1) \otimes r_2(b_2, a_2). \tag{1}$$

This can be interpreted as follows: objects b_1 and b_2 are related if they have attributes that are linked by the external information. If the attribute sets are equal and p is the identity relation then we are naturally looking for a connection between objects such that both have at least one attribute.

Example 1. Consider two formal contexts, the first linking students with the outcomes of their secondary school subjects, and the second linking university study programmes and the scientific fields of these study programmes.

The interpretation of the benevolent method here is as follows: assume that the external information represents how school subjects and scientific fields are related (e.g. biophysics is more related to biology, chemistry and physics and less related to history). Then there is a connection between a student and a programme if the external relation contains at least one of the student's subjects with satisfactory results and one of the scientific orientations of the study programme.

3.2. Rigorous method

Given two contexts $C_1 = (B_1, A_1, r_1)$ and $C_2 = (B_2, A_2, r_2)$, we define a new context $C_2 \to C_1 = (A_1 \times A_2, B_1 \times B_2, r_\to)$ where

$$r_\to((a_1, a_2), (b_1, b_2)) = r_2(b_2, a_2) \to r_1(b_1, a_1).$$

A rigorous bond between C_1 and C_2 corresponding to external information $p \in \mathcal{L}^{A_1 \times A_2}$ can be obtained as the result of $\uparrow p$ computed on $C_2 \to C_1$, that is,

$$\uparrow p(b_1, b_2) = \bigwedge_{(a_1,a_2)\in A_1\times A_2} (p(a_1, a_2) \to (r_2(b_2, a_2) \to r_1(b_1, a_1))) \tag{2}$$

This can be interpreted as a connection between objects b_1 and b_2 in which for all the attributes of b_2, all their connections into attributes of A_1 via p, are included in attributes of b_1.

Example 2. Continuing with our running example, the interpretation of the rigorous method links pairs of students and study programmes (b_1, b_2) where all school subjects related to the scientific orientations of study programme b_2 are included in the set of subjects of student b_1 with satisfying results. Thus, the student is one who is perfectly oriented to the study programme.

4. Theoretical results

This section presents new theoretical results for the described method of connecting two contexts (or concept lattices) given external information, in both rigorous and benevolent approaches.

4.1. Benevolent form

In order to ease the reading, in the statements of the results we will write $\nearrow^b$(resp. $\swarrow^b$) to denote $\nearrow$ (resp. $\swarrow$) in the context $C_1 \otimes C_2$, i.e. the construction in (1).

Theorem 1. *Let $C_i = (B_i, A_i, r_i)$ for $i \in \{1,2\}$ be two $\mathcal{L}$-contexts and $p \in \mathcal{L}^{A_1\times A_2}$ be a fuzzy relation between attribute sets A_1 and A_2. Then for all $(f_1, f_2) \in \mathrm{CL}(B_1, B_2, \nearrow^b p, \nwarrow, \searrow)$ and all $(a_1, a_2) \in A_1 \times A_2$ the following inequality holds*

$$\nearrow_2 f_2(a_2) \to \nwarrow_1 f_1(a_1) \geq p(a_1, a_2) \tag{3}$$

Proof.

$$\begin{aligned}
&\nearrow_2 f_2(a_2) \to \nwarrow_1 f_1(a_1)\\
&\quad= \left(\bigvee_{b_2\in B_2} f_2(b_2)\otimes r_2(b_2,a_2)\right) \to \bigwedge_{b_1\in B_1}(r_1(b_1,a_1)\to f_1(b_1))\\
&\quad= \bigwedge_{b_2\in B_2}\bigwedge_{b_1\in B_1}((f_2(b_2)\otimes r_2(b_2,a_2))\to(r_1(b_1,a_1)\to f_1(b_1)))\\
&\quad= \bigwedge_{b_2\in B_2}\bigwedge_{b_1\in B_1}((f_2(b_2)\otimes r_2(b_2,a_2)\otimes r_1(b_1,a_1))\to f_1(b_1))\\
&\quad= \bigwedge_{b_2\in B_2}\bigwedge_{b_1\in B_1}((r_1(b_1,a_1)\otimes r_2(b_2,a_2))\to(f_2(b_2)\to f_1(b_1)))\\
&\quad\overset{(*)}{\geq} \bigwedge_{b_2\in B_2}\bigwedge_{b_1\in B_1}((r_1(b_1,a_1)\otimes r_2(b_2,a_2))\to\nearrow^b p(b_1,b_2)\\
&\quad= \swarrow_b\nearrow_b p(a_1,a_2)\geq p(a_1,a_2)
\end{aligned}$$

where $(*)$ follows because $(f_1, f_2) \in \mathrm{CL}(B_1, B_2, \nearrow^b p, \nearrow, \swarrow)$ and, as a result, we have $f_2(b_2) \to f_1(b_1) \geq \nearrow^b p(b_1, b_2)$.

Now, we can formulate the following result.

Corollary 1. *Let $C_i = (B_i, A_i, r_i)$ for $i \in \{1,2\}$ be two $\mathcal{L}$-contexts, consider $p \in L^{A_1\times A_2}$ a fuzzy relation, and the context $C_p = (A_1, A_2, p)$. Then, for all $(f_1, f_2) \in \mathrm{CL}(B_1, B_2, \nearrow^b p, \nwarrow, \searrow)$ it holds*

$$\nwarrow_1 f_1 \geq \nearrow_p\nearrow_2 f_2$$

where $\nearrow_p$ is the benevolent concept-forming operator on context C_p.

Proof. From the previous theorem, for all $(a_1, a_2) \in A_1 \times A_2$, we have

$$\begin{aligned}
\nearrow_2 f_2(a_2) \to \nwarrow_1 f_1(a_1) &\geq p(a_1,a_2)\\
\nwarrow_1 f_1(a_1) &\geq p(a_1,a_2)\otimes \nearrow_2 f_2(a_2)\\
\nwarrow_1 f_1(a_1) &\geq \bigvee_{a_2\in A_2} p(a_1,a_2)\otimes \nearrow_2 f_2(a_2)\\
\nwarrow_1 f_1(a_1) &\geq \nearrow_p\nearrow_2 f_2(a_1)
\end{aligned}$$

Example 3. Continuing with our running example, the interpretation of Theorem 2 and Corollary 2 is based on the fact that all subjects connected via p to any of the scientific fields of any of the study programmes of f_2 are included in the characteristic subjects of students from f_1 with satisfying results. This means that for each subject we can find a student from f_1 with satisfying results in the selected subject.

4.2. Rigorous form

In order to ease the reading, in the statements of the results we will write $\uparrow^r$(resp. $\downarrow^r$) to denote $\uparrow$ (resp. $\downarrow$) in the context $C_2 \to C_1$, i.e. the construction in (2).

Theorem 2. *Let $C_i = (B_i, A_i, r_i)$ for $i \in \{1,2\}$ be two $\mathcal{L}$-contexts and $p \in \mathcal{L}^{A_1 \times A_2}$ be an $\mathcal{L}$-relation between the sets of attributes A_1 and A_2. Then for all $(f_1, f_2) \in \mathrm{CL}(B_1, B_2, \uparrow^r p, \uparrow, \downarrow)$ and all $(a_1, a_2) \in A_1 \times A_2$ the following inequality holds*

$$\nearrow_2 f_2(a_2) \to \uparrow_1 f_1(a_1) \geq p(a_1, a_2) \tag{4}$$

Proof.

$$\begin{aligned}
\nearrow_2 (f_2)(a_2) \to \uparrow_1 (f_1)(a_1) & \\
&= \left(\bigvee_{b_2 \in B_2} (f_2(b_2) \otimes r_2(b_2, a_2)) \right) \to \bigwedge_{b_1 \in B_1} (f_1(b_1) \to r_1(b_1, a_1)) \\
&= \bigwedge_{b_1 \in B_1} \bigwedge_{b_2 \in B_2} ((f_2(b_2) \otimes r_2(b_2, a_2)) \to (f_1(b_1) \to r_1(b_1, a_1))) \\
&= \bigwedge_{b_1 \in B_1} \bigwedge_{b_2 \in B_2} ((f_2(b_2) \otimes f_1(b_1)) \to (r_2(b_2, a_2) \to r_1(b_1, a_1))) \\
&\overset{(*)}{\geq} \bigwedge_{b_1 \in B_1} \bigwedge_{b_2 \in B_2} (\uparrow^r p(b_1, b_2) \to (r_2(b_2, a_2) \to r_1(b_1, a_1))) \\
&= \downarrow^r \uparrow^r p(a_1, a_2) \geq p(a_1, a_2)
\end{aligned}$$

where $(*)$ follows because $(f_1, f_2) \in \mathrm{CL}(B_1, B_2, \uparrow^r p, \uparrow, \downarrow)$ hence $(f_1(b_1) \otimes f_2(b_2)) \leq \uparrow^r p(b_1, b_2)$.

Now we can see the conceptual connection between subjects that are (rigorously/strictly) preferred by all students of f_1 and all scientific fields that are (benevolently) associated with one of the study programmes of f_2. The following corollary will further clarify the situation.

Corollary 2. *Let $C_i = (B_i, A_i, r_i)$ for $i \in \{1,2\}$ be two $\mathcal{L}$-contexts. Consider a fuzzy relation $p \in L^{A_1 \times A_2}$, and the context $C_p = (A_1, A_2, p)$. Then for all $(f_1, f_2) \in \mathrm{CL}(B_1, B_2, \uparrow^r p, \uparrow, \downarrow)$ and all $(a_1, a_2) \in A_1 \times A_2$ the following inequality holds*

$$\uparrow_1 f_1 \geq \nearrow_p \nearrow_2 f_2,$$

whereby $\nearrow_p$ is the benevolent concept-forming operator on context C_p.

Proof. From the previous theorem for all $(a_1, a_2) \in A_1 \times A_2$ we have

$$\begin{aligned}
&\nearrow_2 f_2(a_2) \to \uparrow_1 f_1(a_1) \geq p(a_1, a_2) \\
&\quad \text{iff} \quad \uparrow_1 f_1(a_1) \geq p(a_1, a_2) \otimes \nearrow_2 f_2(a_2) \\
&\quad \text{iff} \quad \uparrow_1 f_1(a_1) \geq \bigvee_{a_2 \in A_2} p(a_1, a_2) \otimes \nearrow_2 f_2(a_2) \\
&\quad \text{iff} \quad \uparrow_1 f_1(a_1) \geq \nearrow_p \nearrow_2 f_2(a_1).
\end{aligned}$$

Example 4. The previous results can be interpreted in our running example as follows: For any concept (f_1, f_2), i.e. for any connected group of students f_1 and set of study programmes f_2 it holds that all scientific fields of any of the study programmes $\nearrow_2 f_2$ and any subject connected to any of the study programmes $\nearrow_p \nearrow_2 f_2$ are included in the set of subjects with satisfying results of all students from the group f_1. Thus, all study programmes of f_2 are rigorously well-chosen for the students f_1.

4.3. Benevolent and rigorous interrelationship

In this section, under the assumption of the law of double negation ($\neg\neg k = k$ for all $k \in L$), we recall the equalities corresponding to the inner duality between rigorous and benevolent concept-forming operators.

Let $\nearrow\!\!\!\nearrow$ and $\nwarrow\!\!\!\nwarrow$ be the operators corresponding to the pair of operators $\nearrow$ and $\nwarrow$ defined on the relation $\neg r$. Then we have, on the one hand

$$\uparrow f(a) = \bigwedge_{b \in B} (f(b) \to r(b, a)) = \bigwedge_{b \in B} (\neg r(b, a) \to \neg f(b)) = \nwarrow\!\!\!\nwarrow (\neg f)(a)$$

and, on the other hand,

$$\begin{aligned}
\uparrow f(a) &= \bigwedge_{b \in B} (f(b) \to r(b, a)) = \bigwedge_{b \in B} (f(b) \to \neg\neg r(b, a)) \\
&= \bigwedge_{b \in B} (f(b) \to (\neg r(b, a) \to 0)) = \bigwedge_{b \in B} ((f(b) \otimes \neg r(b, a)) \to 0)
\end{aligned}$$

Table 1
Randomly generated formal contexts C_1, C_2 and external information p.

0.5	0.5	0.5	1	0.5
0.5	0	0.5	1	0.5
0.5	0.5	0	0	0.5
1	1	1	1	0.5
0	1	0.5	0.5	1
1	0.5	0.5	1	0.5

0	1	0	0.5	0.5
0.5	0	0	0	0.5
1	0.5	1	1	0
1	0	0.5	0.5	1
0	1	1	1	1
0.5	0	0	1	0
0.5	0.5	0	0	0.5

0	0.5	0	0.5	0
0.5	0.5	1	0.5	0.5
1	0.5	0.5	1	0.5
1	0.5	1	0.5	0
1	0.5	0.5	0	0

Table 2
Computed bonds $\uparrow^r p = \uparrow_{C_2 \to C_1}(p)$ and $\nearrow^b p = \nearrow_{C_1 \otimes C_2}(p)$.

1	1	0.5	0.5	0.5	0.5	1
0.5	1	0	0.5	0	0.5	1
0.5	0.5	0	0	0	0	0.5
1	1	0.5	0.5	1	1	1
0.5	1	0.5	0.5	0.5	0.5	1
1	1	0.5	0.5	0.5	0.5	1

0.5	0.5	1	1	1	0.5	0.5
0.5	0.5	1	1	1	0.5	0.5
0	0	0.5	0.5	0.5	0	0
0.5	0.5	1	1	1	1	0.5
0.5	0.5	1	1	1	0.5	0.5
0.5	0.5	1	1	1	0.5	0.5

$$= \Big(\bigvee_{b \in B} (f(b) \otimes \neg r(b,a))\Big) \to 0 = \neg \Nearrow f(a)$$

As a result, we obtain

$$\Nwarrow (\neg f)(a) = \neg \Nearrow f(a)$$

5. Experimental study of the upper bounds

In this section, we propose an experimental study of the goodness of the fit provided by the upper bounds on the value of p obtained in the expressions (3) of Theorem 1 and (4) of Theorem 2:

$$\nearrow_2 f_2(a_2) \to \nwarrow_1 f_1(a_1) \geq p(a_1, a_2) \quad \text{for } (f_1, f_2) \in \mathrm{CL}(B_1, B_2, \nearrow^b p, \nwarrow, \searrow)$$

$$\nearrow_2 f_2(a_2) \to \uparrow_1 f_1(a_1) \geq p(a_1, a_2) \quad \text{for } (f_1, f_2) \in \mathrm{CL}(B_1, B_2, \uparrow^r p, \uparrow, \downarrow)$$

The aim is to analyse how tight or not these bounds are, for a variety of situations (different logical structures, different context sizes, etc.). To illustrate the procedure of this experimental study, let us consider the Łukasiewicz logic on a three-valued chain $L = \{0, 0.5, 1\}$, and the two random contexts C_1, C_2 and the external information p shown in Table 1.

These structures can be illustrated by real data as follows. The L-fuzzy formal context C_1 from Table 1 represents six users and their evaluation of five film genres.[1] The second L-fuzzy formal context C_2 from Table 1 illustrates the values of five features of seven films (target audience, budget, tone, topic, and others which are also available in the real datasets). Finally, the external information p corresponds to five features of the five film genres. Regarding the bonds constructed by external information p, we can obtain the degree of recommendation of seven films for six users (Table 2).

Given these formal contexts, the bonds $\uparrow^r p$ and $\nearrow^b p$ are computed according to their definitions in the expressions (1) and (2). The results are shown in Table 2.

In order to check the goodness of the theoretical bounds, we define two operators $p_{\to}$ and $p_{\otimes}$ as follows:

$$p_{\to}(a_1, a_2) = \bigwedge_{(f_1, f_2) \in \mathrm{CL}(B_1, B_2, \uparrow^r p, \uparrow\downarrow)} (\nearrow_2 (f_2)(a_2) \to \uparrow_1 (f_1)(a_1))$$

$$p_{\otimes}(a_1, a_2) = \bigwedge_{(f_1, f_2) \in \mathrm{CL}(B_1, B_2, \nearrow^b p, \nwarrow\searrow)} (\nearrow_2 (f_2)(a_2) \to \nwarrow_1 (f_1)(a_1))$$

Note that each operator is defined as the infimum of the corresponding upper bounds (the left-hand sides of the inequalities proved in the previous section) among all the concepts in the concept lattice of the associated (rigorous or benevolent) bond. The results of this calculation are shown in Table 3. There, we can see that both operators $p_{\to}$ and $p_{\otimes}$ (represented as matrices) are upper approximations to p.

Therefore, to experimentally explore the degree of discrepancy between these matrices and p, we consider as a measure of dissimilarity the normalised sum of the differences between the corresponding matrix values:

$$\mathrm{dis}(p', p) := \frac{1}{|A_1| \cdot |A_2|} \sum_{(a_1, a_2) \in A_1 \times A_2} (p'(a_1, a_2) - p(a_1, a_2)),$$

[1] These types of data are publicly available on Kaggle (https://www.kaggle.com/).

Table 3
Computed matrices $p_{\rightarrow}$, $p_{\otimes}$ and original external information p.

0.5	0.5	0.5	0.5	0.5	1	0.5	1	0.5	1	0	0.5	0	0.5	0
0.5	0.5	1	0.5	0.5	1	0.5	1	0.5	1	0.5	0.5	1	0.5	0.5
1	0.5	1	1	1	1	0.5	1	1	1	1	0.5	0.5	1	0.5
1	0.5	1	1	1	1	0.5	1	0.5	1	1	0.5	1	0.5	0
1	0.5	0.5	0.5	0.5	1	0.5	1	0.5	1	1	0.5	0.5	0	0

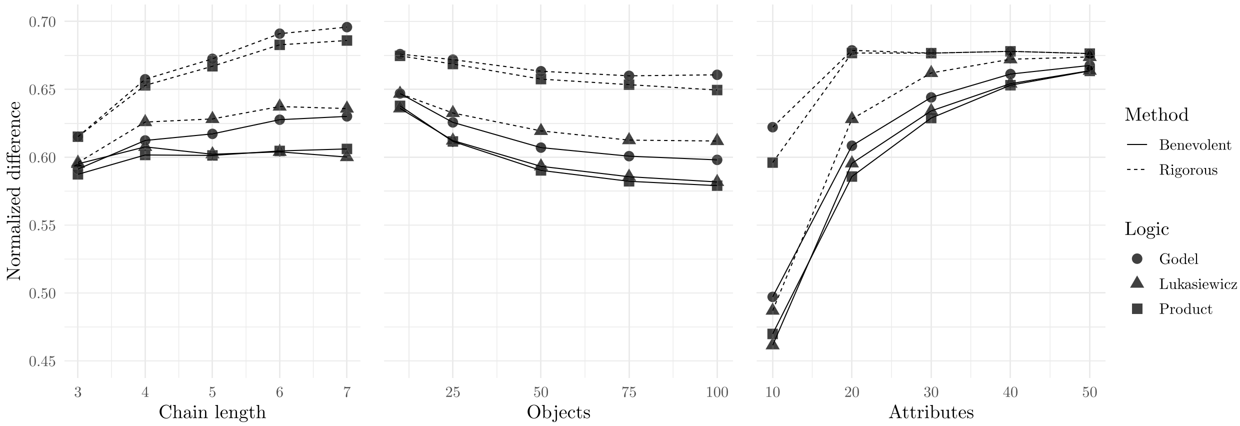

Fig. 2. Representation of the normalised average differences between the upper bounds and the corresponding external information p used in the experiments.

where $p' \in \{p_{\rightarrow}, p_{\otimes}\}$. In particular, we can check in Table 3 that the difference between $p_{\rightarrow}$ and p is $\frac{5}{25} = \frac{1}{5}$ and difference between $p_{\otimes}$ and p is $\frac{8}{25}$.

Following this strategy, we carried out extensive experiments considering formal contexts of dimension $\mathrm{obj} \times \mathrm{att}$, with $\mathrm{obj} = |B_1| = |B_2| \in \{10, 25, 50, 75, 100\}$ and $\mathrm{att} = |A_1| = |A_2| \in \{10, 20, 30, 40, 50\}$). For each dimension, one hundred pairs of formal contexts C_1 and C_2 were randomly generated, together with an external relation of dimension $\mathrm{att} \times \mathrm{att}$. The above inequalities were studied by considering three different logics (Łukasiewicz, Gödel, and product) and five different sets of truth-values (n-valued linear chains for $n \in \{3, 4, 5, 6, 7\}$).

The whole implementation was done in the programming language R and the `fcaR` library [15] was used to compute the bonds, the corresponding concept lattices and the matrices $p_{\rightarrow}$ and $p_{\otimes}$ and to compare them with the value of p using the dissimilarity described above.

The plots in Fig. 2 show the average value of the normalised difference according to the number of attributes (right plot), the number of objects (middle plot) or the length of the chain (left plot), always grouped by method and logic. For example, if the X axis represents the length of the chain, the differences are averaged for each value of the length of the chain for all combinations of the number of objects and the number of attributes.

We can observe a clear trend in each of the plots: (1) as the number of truth-values increases, so does the normalised difference; (2) the difference tends to decrease with a larger number of objects, even suggesting the existence of an asymptotic lower bound on dissimilarity; and (3), the opposite behaviour is seen with respect to the dependence on the number of attributes, as the difference increases with the number of attributes, with a similar potential asymptotic upper bound.

For the sake of completeness, we present in Table 4 some of the exact results obtained for seven truth values, for all logics and context sizes. Note that the trends observed in Fig. 2 can also be seen in Table 4, although there are artefacts due to the smaller number of trials for this exact configuration.

In summary, the proposed experiments and exploration provide the (graphical) demonstration that the connection of all extents f_1 and f_2 of the input contexts C_1 and C_2 is well proposed and corresponds to the external information given in p. It is easy to see how $p_{\rightarrow}$ and $p_{\otimes}$ approximate p with an increasing number of objects. In our problem setting, this implies the natural idea that as the input data becomes richer and larger, the connections will become more precise. However, studying the asymptotic behaviour of the upper bounds in our future work can provide a more detailed insight into the bonds and their properties.

Algorithmic complexity analysis

We start by describing some results corresponding to the algorithmic complexity analysis of related constructions. We divide the analysis into the three main parts of the computations: the principal contribution of this work, the construction of the bonds, is presented first. The other two analyses, for the computation of the concept lattice and the estimation of the discrepancy with the external information, are also presented here for the sake of completeness:

Construction of the bond. Babin and Kuznetsov [3] presented the algorithmic complexity analysis for shared intents. For sets G, M, and for a crisp binary relation $I \subseteq G \times M$, they considered a classical formal context (G, M, I), and defined a *shared intent* for two

Table 4
Normalized averaged differences computed for the case of chain length equal to 7, that is, $L = [0,1]_6 = \{0, 1/6, \ldots, 5/6, 1\}$.

Logic	$\|A\|$	Rigorous					Benevolent				
		10	25	50	75	100	10	25	50	75	100
Godel	10	0.686	0.681	0.626	0.625	0.620	0.606	0.531	0.494	0.470	0.461
	20	0.707	0.710	0.701	0.709	0.708	0.671	0.629	0.603	0.596	0.591
	30	0.707	0.712	0.697	0.707	0.713	0.687	0.672	0.639	0.639	0.637
	40	0.713	0.710	0.705	0.706	0.713	0.700	0.686	0.668	0.666	0.662
	50	0.709	0.703	0.707	0.716	0.706	0.703	0.691	0.686	0.688	0.676
Lukasiewicz	10	0.543	0.482	0.435	0.435	0.424	0.516	0.444	0.405	0.392	0.385
	20	0.672	0.653	0.629	0.609	0.598	0.649	0.612	0.574	0.566	0.549
	30	0.695	0.691	0.678	0.673	0.675	0.679	0.655	0.627	0.627	0.619
	40	0.707	0.703	0.691	0.690	0.700	0.694	0.669	0.648	0.647	0.649
	50	0.707	0.699	0.701	0.710	0.697	0.701	0.684	0.675	0.675	0.661
Product	10	0.671	0.651	0.567	0.571	0.543	0.562	0.494	0.461	0.448	0.441
	20	0.707	0.708	0.701	0.701	0.705	0.655	0.599	0.570	0.562	0.557
	30	0.707	0.712	0.697	0.707	0.713	0.677	0.649	0.608	0.604	0.601
	40	0.713	0.710	0.705	0.706	0.713	0.695	0.673	0.644	0.637	0.630
	50	0.709	0.703	0.707	0.716	0.706	0.700	0.685	0.672	0.672	0.656

formal contexts (G_1, M, I_1) and (G_2, M, I_2) as a set $A \subseteq M$ such as A is intent for (G_1, M, I_1) and (G_2, M, I_2). They proved that the closure of a set $X \subseteq M$ can be solved in $O((|G_1| + |G_2|) \cdot |M|)$ time.

Moreover, they proved that bonds between crisp formal contexts can be represented in terms of shared intents and that the closure operator for bonds of crisp formal contexts (G_1, M_1, I_1) and (G_2, M_2, I_2) can be determined in time $O((|G_1| \cdot |G_2| + |M_1| \cdot |M_2|) \cdot |M_2| \cdot |G_1|)$, given by contranominal scales, i.e. $(G_1, G_1, \neq)$ and $(M_2, M_2, \neq)$.

In our paper, we computed the bonds of $\mathcal{L}$-fuzzy formal contexts according to their definitions in the expressions (1) and (2). We consider external information p and two $\mathcal{L}$-fuzzy formal contexts (B_1, A_1, r_1), (B_2, A_2, r_2). If we assume that the operations $\otimes : L \times L \longrightarrow L$ and $\rightarrow : L \times L \longrightarrow L$ can be computed in $O(1)$ time, both bonds $\uparrow^r p$ and $\nearrow^b p$ can be determined in $O(|B_1| \cdot |B_2| \cdot |A_1| \cdot |A_2|)$ time, since we compute one direct product of two L-fuzzy formal contexts in expressions (1) and (2).

Computation of the concepts. Ganter's algorithm `NextClosure` [16] and Lindig's algorithm `NextNeighbor` [31] are well-known methods for the computation of the concepts and the concept lattice. For a formal context (G, M, I), the asymptotic worst-case complexity of these algorithms is $O(n \cdot |G|^2 \cdot |M|)$, where n is the number of formal concepts.

For the theoretical bounds given by the two operators $p_\rightarrow$ and $p_\otimes$, we need to compute all $\mathcal{L}$-fuzzy formal concepts in the concept lattice of the associated bond. In the worst case, the number of $\mathcal{L}$-fuzzy formal concepts of $\mathrm{CL}(B_1, B_2, \uparrow^r(p), \uparrow\downarrow)$ or $\mathrm{CL}(B_1, B_2, \nearrow^b(p), \nwarrow\searrow)$ is exponential. Hence, the worst-case time complexity of any algorithm for the computation of the concepts is exponential, and we can only expect that these algorithms have a short runtime in the majority of practical situations.

Several approaches have been defined to compute the set of $\mathcal{L}$-concepts, such as the version of the `NextClosure` algorithm [7] for $\mathcal{L}$-fuzzy formal contexts, which generates concepts following the lectic order of their intents, or the generalized `Close-by-One` and Lindig's `NextNeighbor` algorithms [10], which use incremental strategies for the computations. The most complete of these algorithms is Lindig's, since it is able to compute the complete concept hierarchy, that is, the concept lattice, without any further post-processing.

Estimation of the dissimilarity to the external information. Finally, the degree of discrepancy between matrices of two operators $p_\rightarrow$, $p_\otimes$ and external information p can be computed in $O(|A_1| \cdot |A_2|)$ time.

6. Conclusion

In this paper, we have extended the study of two novel methods for selecting appropriate bonds between $\mathcal{L}$-fuzzy formal contexts. Specifically, we have presented the interpretation of the rigorous and benevolent concept-forming operators, as well as the formal verification that the bonds induced by the presence of external information about the connection between attributes produce coherent results with this information.

In particular, the two main theoretical results proposed (Theorems 1 and 2) establish a relationship between the formal concepts derived from the bonds and the external information p used in their construction: their extents and intents allow us, in a sense, to determine an upper bound (i.e., the matrices obtained by operators $p_\rightarrow$ and $p_\otimes$) for external information p. Since the bonds are intended (by design) to implicitly incorporate the information contained in p (representing the connection between attributes), these theoretical results confirm that the knowledge in p is effectively contained in the concept lattice of the bond.

To complete the study, experimental tests were conducted to verify the fit of these bounds to the original value of the external information provided. Using the R programming language, pairs of random $\mathcal{L}$-fuzzy formal contexts were generated with varying numbers of objects and attributes, repeating the process of constructing the bond 100 times for each size (also randomly generating the external information). In this way it has been possible to verify that the upper bounds obtained in Theorems 1 and 2 show some interesting behaviour: as the number of objects increases, the computed difference between the upper bounds and the information p

decreases, although this trend is reversed when the number of attributes increases, since it can be verified that in this situation the *dissimilarity* between p and the upper bounds also increases. These results, obtained using different underlying logics (Łukasiewicz, Gödel and product), confirm the idea that, with a large number of objects, the constructed bonds will be more precise in representing the additional information required.

This leads us to propose as future research the study of the asymptotic behaviour of the upper bounds, in relation to several variables – the number of attributes, the number of objects, and the length of the truth value chain – as well as other possible theoretical developments as a continuation of this work.

CRediT authorship contribution statement

Ondrej Krídlo: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Validation, Writing – review & editing. **Domingo López-Rodríguez:** Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Validation, Writing – review & editing. **Lubomir Antoni:** Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Validation, Writing – review & editing. **Peter Eliaš:** Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Validation, Writing – review & editing. **Stanislav Krajči:** Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Validation, Writing – review & editing. **Manuel Ojeda-Aciego:** Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Validation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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