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Analysis of Fuzzy Vector Spaces as an Algebraic Framework for Flag Codes

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Abstract: Flag codes are a recent network coding strategy based on linear algebra. Fuzzy vector subspaces extend the notions of classical linear algebra. They can be seen as abstractions of flags to the point that several fuzzy vector subspaces can be identified to the same flag, which naturally induces an equivalence relation on the set of fuzzy vector subspaces. The main contributions of this work are the methodological abstraction of flags and flag codes in terms of fuzzy vector subspaces, as well as the generalisation of three distinct equivalence relations that originated from the fuzzy subgroup theory and study of their connection with flag codes, computing the number of equivalence classes in the discrete case, which represent the number of essentially distinct flags, and a comprehensive analysis of such relations and the properties of the corresponding quotient sets.

Keywords: fuzzy vector spaces; level sets; similarity relation; isomorphism relation; flag codes

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1. Introduction

The process of classifying mathematical objects holds significant importance in mathematics, in the case of, for instance, finite groups [1,2], simple groups [3] or topological spaces [4]. It serves as a vital tool when the aim is to compare, analyse and gain deeper insight into these structures. Equivalence relations are useful for classifying mathematical objects with similar properties. Classifying mathematical objects establishes structured partitions, simplifying the analysis of algebraic structures by identifying equivalent classes. Moreover, it is essential for constructing more complex mathematical structures, enabling the definition of new structures through the identification of objects that are "equal" under some conditions.

It is crucial to emphasise that the significance of classifications transcends theoretical interest, finding practical applicability in domains such as those motivating this work: information theory [5] and the investigation of flag codes within network coding.

In the realm of network coding [6], flag codes [7,8] represent a category of error-correcting codes deployed to safeguard data packets from corruption, ensuring dependable communication. Their efficacy in network coding stems from their efficient encoding and decoding capabilities, even in the presence of multiple data flows.

A flag code is defined as a family of nested subspaces within a vector space, each representing distinct degrees of error correction that the code can provide. Building upon this motivation, it becomes evident that the abstraction of flags and flag codes finds manifestation in the concept of fuzzy vector subspaces [9], which will be expounded shortly. The classes of equivalence inherent in this classification framework offer a means to define "codes" that are essentially identical. Consequently, the exploration of the diverse array of distinct flag codes (without considerations of optimality) that could be contemplated in a given problem emerges as a captivating avenue of study.

The purpose of this paper is to analyse equivalence relations on fuzzy vector spaces as an extended view of flags (and flag codes, which serve as the motivation for the analysis) and their relations from an algebraic standpoint. The study of fuzzy algebraic structures originated soon after the first appearance of Fuzzy Set Theory [10]. Since then, several authors have devoted their efforts to examining the properties of fuzzy algebras [11], fuzzy subgroups [12,13], fuzzy vector spaces [14,15], fuzzy rings [16] or fuzzy semirings [17]. In the context of fuzzy vector spaces (introduced in [14]), the focus is on understanding the distinct fuzzy vector subspaces that can be defined within a vector space. In order to classify fuzzy vector subspaces within a finite-dimensional vector space E, denoted as $\mathcal{F}(E)$, in this work, various partitions on $\mathcal{F}(E)$ are examined based on the following criteria:

1. The partition on $\mathcal{F}(E)$ determined by an isomorphism relation.
2. The partition on $\mathcal{F}(E)$ established by the level sets.
3. The partition on $\mathcal{F}(E)$ resulting from the combination of the preceding two criteria.

All the results are presented in this context, meaning, how many possible classifications are there is determined via each of the three partitions above. Note that these relations can be seen as the mathematical expression for the *equality* of flags used in flag codes, in the network coding terminology.

The isomorphism relation has proven valuable in various algebraic structures, including fuzzy sets [18,19], fuzzy graphs [20], or fuzzy rings and fuzzy ideals [17]. Level sets, highlighted as a valuable tool within Fuzzy Set Theory, facilitate the classification of fuzzy algebraic structures based on distinct cuts. Further information can be found in [21–24]. Combining both relations, a new way of classifying mathematical objects is obtained [25,26].

Whenever an equivalence relation arises, it is a natural step to consider its quotient set, that is, the set of equivalence classes, and examine whether this quotient inherits the properties of the main structure. There are relevant examples in the literature, for instance, the quotient by any vector subspace can be computed and the quotient set will be a vector space. However, in groups, the quotient needs to be defined on a normal subgroup since the quotient via arbitrary subgroups does not behave properly; the case of rings is similar, where only the quotient via ideals behaves properly, but not for general subrings [27]. The topic of interest in this paper is to examine whether the quotient sets of three relevant equivalence relations in the study of fuzzy subgroups behave well in the study of fuzzy vector subspaces.

The main contributions and ideas of this paper can be summarised as follows:

- Methodological abstraction of flags and flag codes in the terminology of fuzzy algebraic structures: This paper establishes a comprehensive link between flags used in flag codes within network coding and fuzzy vector spaces, bridging the conceptual gap between these two seemingly distinct algebraic structures. The concept of fuzzy vector subspaces is introduced as an expression of the abstraction of flags, providing a novel perspective for analysis and classification.
- Generalisation of equivalence relations in fuzzy algebraic structures: This article expands on the use of equivalence relations, which have been historically valuable in various algebraic structures such as fuzzy sets, fuzzy graphs, fuzzy rings and fuzzy ideals, by applying them to fuzzy vector spaces. This generalisation allows for the exploration of structural similarities and classifications within fuzzy algebraic structures, contributing to a broader understanding of mathematical equivalence in these contexts.
- Algebraic analysis of equivalence relations: This article presents an algebraic analysis of equivalence relations on fuzzy vector spaces, with a focus on three primary equivalence relations: those determined by isomorphism, level sets and their combination. The study provides a detailed examination of how these relations influence the classification and quotient sets of fuzzy vector subspaces. This analysis extends the understanding of the mathematical expression for the "equality" of flags used in flag codes within network coding.

- Enumeration of equivalence classes: This paper provides a detailed list of the different equivalence classes that arise from each equivalence relation. This analysis clarifies the subtle differences and similarities between fuzzy vector subspaces, offering insights into the complex relationships between different classes. The enumeration highlights the practical implications for flag codes, demonstrating how these different classes affect the interchangeability and practical application of flags in the network coding framework.
- Quotient set analysis: This paper examines the behaviour of quotient sets resulting from three essential equivalence relations in the study of fuzzy vector subspaces. The investigation aims to determine whether the quotient sets preserve the properties of the main structure, drawing parallels with established examples in group theory and ring theory. This analysis enhances the understanding of the algebraic properties inherited by quotient sets in the context of fuzzy vector subspaces.

This paper is structured as follows: Section 2 recalls the main ideas behind the use of flag codes in network coding. Section 3 remembers the basic notions to understand the results of the article. Section 4 is devoted to showing the three equivalence relations and examples of fuzzy vector subspaces in each partition. The main results are in Section 5, which presents how the fuzzy vector subspaces are connected considering the three equivalences. Section 6 is focused on the analysis of the algebraic operations (infimum and sum) on the respective quotient sets. Finally, Section 7 provides some concluding remarks and future works.

2. Flag Codes in Network Coding

In this section, some basic aspects of network coding and the use of flag codes are presented, which is the motivation behind the algebraic analysis performed in the rest of this paper. Introduced in [6], network coding has emerged as a transformative strategy for enhancing information flow within a network characterised by an acyclic-directed graph, potentially accommodating multiple sources and destinations. Unlike conventional routing-based approaches, network coding employs a sophisticated algebraic approach, utilising vector spaces over finite fields. In particular, the network in consideration functions with vectors from a specified vector space $\mathbb{F}_q^n$ over the finite field of q elements $\mathbb{F}_q$, where q denotes a prime power.

Koetter and Kschischang established the algebraic foundations of network coding in their seminal paper [28]. They recognised the inherent invariance of vector spaces under linear combinations, which led to using vector subspaces as codewords. This move away from single vectors as codewords has several advantages, allowing more efficient and reliable data transmission. The network works by manipulating data streams at intermediate nodes rather than simply routing them, leading to significant performance gains: the sender injects the vector space into the network, and each intermediate node performs coding by generating random linear combinations of the received vectors. The receiver, upon receiving these signals, seamlessly reconstructs the original vector space by forming the $\mathbb{F}_q$ vector subspace spanned by the incoming vectors.

In this context, a subspace code of length n is essentially a non-empty collection of subspaces of $\mathbb{F}_q^n$. The utilisation of subspace codes necessitates a single use of the aforementioned channel to convey a codeword, specifically, a subspace. This concept was later extended in [29] into the domain of multishot subspace codes. To be more precise, in an r-shot code, codewords manifest as sequences comprising $r \geq 2$ vector subspaces of $\mathbb{F}_q^n$. In this scenario, the transmission of a codeword requires r uses (shots) of the channel.

Flag codes, a distinctive subclass of subspace codes, have gained prominence due to their hierarchical structure and enhanced efficiency. Introduced in [7], flag codes are characterised by codewords consisting of sequences of nested subspaces (flags). This nested structure offers a more organised representation compared to general subspace codes, paving the way for more sophisticated network coding schemes.

For integers $1 \leq t_1 < t_2 < \ldots < t_r < n$, a *flag*, denoted by $\mathcal{L}$, of *type* (or *signature*) $(t_1, t_2, \ldots, t_r)$ on $\mathbb{F}_q^n$ comprises a sequence of nested vector subspaces:

$$U_1 \subsetneq U_2 \subsetneq \ldots \subsetneq U_r \subsetneq \mathbb{F}_q^n,$$

where $\dim U_i = t_i$. The t_i-dimensional subspace U_i is designated as the i-th subspace of the flag $\mathcal{L}$. Full flags, specifically flags of the type $(1, \ldots, n-1)$, are particularly noteworthy and represent the highest degree of nesting.

The flag variety of type $(t_1, t_2, \ldots, t_r)$ on $\mathbb{F}_q^n$ is symbolised as $\mathcal{L}_q((t_1, t_2, \ldots, t_r), n)$, representing the set encompassing flags of the corresponding type. A *flag code* of type $(t_1, t_2, \ldots, t_r)$ on $\mathbb{F}_q^n$ is a non-empty subset D of the flag variety $\mathcal{L}_q((t_1, t_2, \ldots, t_r), n)$.

3. Preliminaries

This paper aims to analyse and classify fuzzy vector subspaces based on different equivalence relations, as abstract and general expressions of the flags and flag codes mentioned. The theoretical results of the study will be presented in successive sections, followed by a discussion of their implications on the algebraic structure of the variety of flags and flag codes.

Throughout the paper, let E represent a vector space that operates within a field denoted as $\mathbb{F}$, and let 0 and 1 designate the neutral elements for the inner operations within the field $\mathbb{F}$.

Definition 1 ([14])**.** *Let $(E, +, \cdot)$ be a vector space on a field $\mathbb{F}$ and $\mu : E \longrightarrow [0,1]$ a fuzzy set of E. The mapping μ is said to be a fuzzy vector subspace of E if it satisfies the following axioms:*

$(E1)$ $\mu(x+y) \geq \mu(x) \wedge \mu(y)$ *for all* $x, y \in E$.
$(E2)$ $\mu(ax) \geq \mu(x)$ *for all* $x \in E$ *and* $a \in \mathbb{F}$.

The set of all fuzzy vector subspaces of E will be denoted by $\mathcal{F}(E)$. Das, in [30], investigated the characteristics of fuzzy vector spaces. They introduced a departure from the conventional use of the minimum operator in axiom $(E1)$, opting instead for an arbitrary t-norm.

Following Definition 1, it can be inferred that $\mu(\mathbf{0}) \geq \mu(x)$ holds for every element $x \in E$, where $\mathbf{0}$ represents the neutral element of the abelian group $(E, +)$. Below, a characterisation of a fuzzy vector subspace in terms of its level sets is shown; this will play a crucial role in this article.

Definition 2 ([10])**.** *Let $(E, +, \cdot)$ be a vector space on a field $\mathbb{F}$ and $\mu : E \longrightarrow [0,1]$ a fuzzy set of E. For each $t \in [0,1]$, the level set μ_t is defined as follows:*

$$\mu_t = \{x \in E \mid \mu(x) \geq t\}.$$

A fuzzy vector subspace is characterised by its level sets.

Proposition 1 ([15])**.** *Let $\mu : E \longrightarrow [0,1]$ be a fuzzy set of a vector space E on a field $\mathbb{F}$. The following assertions are equivalent:*

1. *μ is a fuzzy vector subspace of E.*
2. *$\mu(ax+by) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in E$ and $a, b \in \mathbb{F}$.*
3. *Each non-empty level set of μ is a vector subspace of E.*

The level sets of a fuzzy vector subspace are a chain of subspaces of E,

$$\mu_{t_1} \subseteq \mu_{t_2} \subseteq \mu_{t_3} \subseteq \cdots \subseteq \mu_{t_i} \subseteq \mu_{t_{i+1}} \subseteq \ldots$$

where $t_i > t_{i+1}$ for all $i \in \mathbb{N}$. Since $\mu_0 = E$, the supreme of the chain is E. Hence, in a space E, a fuzzy vector subspace is then characterised by a (possibly infinite) sequence

$$\{(U_0, t_0), (U_1, t_1), \ldots, (E, 0)\}$$

such that $U_i = \mu_{t_i}$ for all i, with $t_i > t_{i+1}$. For ease of representation, and without loss of generality, only sequences where $U_i \subsetneq U_{i+1}$ for all i will be considered in this work. These can be constructed just by an appropriate selection of t_i.

This sequence, if $U_0 \neq \{\mathbf{0}\}$, corresponds to a *flag* of a type determined by the respective dimensions of U_i. In the rest of the paper, chains of subspaces where the first one is not necessarily non-null will be considered, but all of them will end in E. Thus, let an *extended flag* be the sequence of subspaces where $U_0 = \{\mathbf{0}\}$ and $U_m = E$, under the same conditions of monotony as above. Therefore, a fuzzy vector subspace can be seen as a generalisation of a flag.

Let us present two operations on fuzzy vector subspaces, mimicking the *intersection* and *sum* of flags, which play significant roles in the crisp context. Their fuzzification was introduced in [31] as follows:

Definition 3 ([31])**.** *Let $(E, +, \cdot)$ be a vector space on a field $\mathbb{F}$ and $\mu, \eta : E \longrightarrow [0, 1]$ two fuzzy vector subspaces of E.*

1. *The intersection, or minimum, $\mu \wedge \eta : E \longrightarrow [0, 1]$ is defined by*

$$(\mu \wedge \eta)(x) := \mu(x) \wedge \eta(x).$$

2. *The sum $\mu + \eta : E \longrightarrow [0, 1]$ is defined using Zadeh's extension principle [32],*

$$(\mu + \eta)(x) := \sup\{\mu(y) \wedge \eta(z) \mid x = y + z\}.$$

In the crisp case, both operations induce vector subspaces. It has already been proved that the same situation happens in the fuzzy paradigm.

Proposition 2 ([31])**.** *Let $(E, +, \cdot)$ be a vector space on a field $\mathbb{F}$ and $\mu, \eta : E \longrightarrow [0, 1]$ two fuzzy vector subspaces of E. Then, the minimum $\mu \wedge \eta$ and the sum $\mu + \eta$ are fuzzy vector subspaces of E.*

In Section 6, the effects of these two operations on the quotient sets induced by the equivalence relations are studied.

Note that fuzzy vector subspaces extend flags in the sense that different fuzzy vector subspaces may induce the same (extended) flag. This is shown in the following example.

Example 1. *Let us consider a finite field $\mathbb{F}$ with q elements. Let us define the fuzzy sets $\mu, \eta : \mathbb{F}^2 \longrightarrow [0, 1]$ as follows:*

$$\mu(x, y) = \begin{cases} 0.6 & \text{if} \quad x = 0, \\ 0.2 & \quad\text{otherwise,} \end{cases}$$

$$\eta(x, y) = \begin{cases} 0.7 & \text{if} \quad x = 0, \\ 0.3 & \quad\text{otherwise.} \end{cases}$$

It is immediate to see that they are fuzzy vector subspaces of $\mathbb{F}^2$. Moreover, the level sets of μ are $\mu_{0.6}$ and $\mathbb{F}^2$, which are 1-dimension and 2-dimension spaces. Therefore, μ induces a flag, that is, a sequence of two crisp subspaces of $\mathbb{F}^2$. Analogously, η induces the same flag since $\mu_{0.6} = \eta_{0.3}$.

Therefore, it is interesting to define algebraic relations between them to describe the correspondence between these two types of entities. In this work, equivalence relations are considered since they express the idea of the "equality" of flags and flag codes formally. That is, equivalence relations on fuzzy vector subspaces allow us to express the conditions where two flags or flag codes are *interchangeable* from a practical point of view, and the use of one or another makes no difference.

Let us begin with two equivalence relations introduced in the context of fuzzy subgroups.

Definition 4 ([33]). *Let $(E, +, \cdot)$ be a vector space and $\mu, \eta : E \longrightarrow [0, 1]$ two fuzzy vector subspaces. The fuzzy vector subspace μ is said to be isomorphic to η (denoted $\mu \cong \eta$) if there is an isomorphism $f : E \longrightarrow E$ satisfying that*

$$\mu(x) = \eta(f(x)), \tag{1}$$

for all $x \in E$.

Definition 5 ([21]). *Let $(E, +, \cdot)$ be a vector space and $\mu, \eta : E \longrightarrow [0, 1]$ two fuzzy vector subspaces. The fuzzy vector subspace μ is said to be similar to η (denoted $\mu \sim \eta$) if*

$$\{\mu_t \mid t \in \mathrm{Im}\, \mu\} = \{\eta_s \mid s \in \mathrm{Im}\, \eta\}, \tag{2}$$

for all $x \in E$.

These relations allow us to identify flags whose characteristics are identical:

- When two fuzzy vector subspaces verify $\mu \cong \eta$, there is an isomorphism $f : E \to E$ that transforms the extended flag associated to μ into the one associated to η: since $\mu(x) = y$ if, and only if, $\eta(f(x)) = y$, thus a subspace U is the α-cut of μ if, and only if, $f(U)$ is the α-cut of η. That is, if $\mu \cong \eta$, then
$$\mu \equiv \{(U_0, t_0), (U_1, t_1), \ldots\},$$
and
$$\eta \equiv \{(f(U_0), t_0), (f(U_1), t_1), \ldots\},$$
for a given automorphism f on E.
- The similarity relation is more explicit since the chains of subspaces associated with μ and η, if $\mu \sim \eta$, are completely identical, but the cuts in both subspaces may not coincide. That is, if $\mu \sim \eta$, then
$$\mu \equiv \{(U_0, t_0), (U_1, t_1), \ldots\},$$
and
$$\eta \equiv \{(U_0, s_0), (U_1, s_1), \ldots\},$$
for possibly distinct $\{t_i\}$ and $\{s_i\}$.

The following characterization will be useful for understanding examples and proofs.

Proposition 3 ([34]). *Let $(E, +, \cdot)$ be a vector space and $\mu, \eta : E \longrightarrow [0, 1]$ two fuzzy vector subspaces. Then, μ is similar to η if and only if*

$$\mu(x) < \mu(y) \iff \eta(x) < \eta(y) \tag{3}$$

for all $x, y \in E$.

4. Equivalence Relations on Fuzzy Vector Spaces

The following novel equivalence relation is a combination of the previous two and arises naturally from the study of fuzzy subgroups [26].

Definition 6. *Let $(E, +, \cdot)$ be a vector space and $\mu, \eta : E \longrightarrow [0,1]$ two fuzzy vector subspaces. The fuzzy vector subspace μ is said to be iso-similar to η (denoted $\mu \simeq \eta$) if there is an isomorphism $f : E \longrightarrow E$ such that*

$$\{\mu_t \mid t \in \text{Im } \mu\} = \{f(\eta_s) \mid s \in \text{Im } \eta\}.$$

A useful characterisation of iso-similar fuzzy vector subspaces is given in the next result:

Proposition 4. *Let $(E, +, \cdot)$ be a vector space and $\mu, \eta : E \longrightarrow [0,1]$ two fuzzy vector subspaces. Then, μ is iso-similar to η if and only if there is an isomorphism $f : E \longrightarrow E$ satisfying*

$$\mu(x) < \mu(y) \iff \eta(f(x)) < \eta(f(y)), \tag{4}$$

for all $x, y \in E$.

Proof. We know that μ is iso-similar to η if and only if there is an isomorphism $g : E \longrightarrow E$ such that

$$\{\mu_t \mid t \in \text{Im } \mu\} = \{g(\eta_s) \mid s \in \text{Im } \eta\}.$$

Considering that g is an isomorphism, notice that

$$\begin{aligned} g(\eta_s) &= \{y \in E \mid \exists x \in \eta_s \text{ with } g(x) = y\} \\ &= \{y \in E \mid g^{-1}(y) \in \eta_s\} \\ &= \{y \in E \mid \eta(g^{-1}(y)) > s\}. \end{aligned}$$

Based on Proposition 3 and putting $f = g^{-1}$, we conclude that μ is iso-similar to η if and only if there is an isomorphism $f : E \longrightarrow E$ satisfying

$$\mu(x) < \mu(y) \iff \eta(f(x)) < \eta(f(y)),$$

for all $x, y \in E$. □

Notation 1. *Given a fuzzy vector subspace μ of E, its corresponding equivalence classes induced by the three relations (isomorphism, similarity, and iso-similarity, respectively) will be denoted by $[\mu]_{\cong}$, $[\mu]_{\sim}$, and $[\mu]_{\simeq}$. In addition, the usual notation for quotient sets will be used throughout this work; that is, the three induced quotient sets will be denoted by $\mathcal{F}(E)/\simeq$, $\mathcal{F}(E)/\cong$, and $|\mathcal{F}(E)/\sim$.*

In terms of flags, this relation can be seen as an extension of both the isomorphism and the similarity relations. Note that, when $\mu \simeq \eta$, there exists an automorphism f on E such that $\{\eta_s\}_{s \in [0,1]} = \{f(\mu_t)\}_{t \in [0,1]}$. Then, the flags associated with them can be written as follows:

$$\mu \equiv \{(U_0, t_0), (U_1, t_1), \ldots\},$$

and

$$\eta \equiv \{(f(U_0), s_0), (f(U_1), s_1), \ldots\},$$

where the symbol $\equiv$ has been used to signify the characterisation of a fuzzy vector subspace using an extended flag.

As a matter of fact, the iso-similarity relation is finer than the two defined above. This is shown in the next result.

Proposition 5. *Let E be a vector space on a field $\mathbb{F}$ and $\mu, \eta : E \longrightarrow [0,1]$ two fuzzy vector subspaces.*

1. *If μ is isomorphic to η, then μ is iso-similar to η.*
2. *If μ is similar to η, then μ is iso-similar to η.*

Proof. If μ is isomorphic to η, there is an isomorphism $f : E \longrightarrow E$ satisfying

$$\mu(x) = \eta(f(x))$$

for all $x \in E$. Since $\mu = f(\eta)$, therefore, the level sets of μ and $f(\eta)$ are equal. Consequently, μ is iso-similar to η. If μ is similar to η, they have the same level sets. Considering the identity function ι on E, which is an isomorphism, hence, μ is iso-similar to η. □

In other terms, if $\eta \in [\mu]_{\cong}$, based on Proposition 5, then $\eta \in [\mu]_{\simeq}$; therefore, $[\mu]_{\cong} \subseteq [\mu]_{\simeq}$. By analogous reasoning, the result $[\mu]_{\sim} \subseteq [\mu]_{\simeq}$ also holds. Hence,

Corollary 1. *Let E be a vector space. Then*

$$\left|\mathcal{F}(E)/\simeq\right| \leq \min\{\left|\mathcal{F}(E)/\cong\right|, \left|\mathcal{F}(E)/\sim\right|\}.$$

Below, some examples of fuzzy vector subspaces that are connected using these three equivalence relations are presented.

Example 2. *Let us consider the real vector space $\mathbb{R}^2$ and two different vector lines $r_1, r_2 \subseteq \mathbb{R}^2$. Let us define the fuzzy sets $\mu, \eta, \nu : \mathbb{R}^2 \longrightarrow [0,1]$ as follows:*

$$\mu(u) = \begin{cases} 0.7 & \text{if} \quad u \in r_1, \\ 0.4 & \quad\quad \text{otherwise}, \end{cases}$$

$$\eta(u) = \begin{cases} 0.7 & \text{if} \quad u \in r_2, \\ 0.4 & \quad\quad \text{otherwise}, \end{cases}$$

$$\nu(u) = \begin{cases} 0.5 & \text{if} \quad u \in r_1, \\ 0.2 & \quad\quad \text{otherwise}. \end{cases}$$

It is easy to check that they are fuzzy vector spaces using Proposition 1. The three equivalence relations provide the following connections.

1. *Consider any isomorphism $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ satisfying $f(r_1) = r_2$. This is possible because the vector lines have the same dimension. Clearly,*

$$\mu(f(u)) = \eta(u),$$

for all $u \in \mathbb{R}^2$. Then, μ is isomorphic to η. However, both of them cannot be isomorphic to ν because their ranges of values are different to $\operatorname{Im}\nu = \{0.2, 0.5\}$.
2. *Notice that the level sets of the three fuzzy vector spaces are*

$$\begin{aligned} \{\mu_t\}_{t \in \operatorname{Im}\mu} &= \{r_1, \mathbb{R}^2\}, \\ \{\eta_t\}_{t \in \operatorname{Im}\eta} &= \{r_2, \mathbb{R}^2\}, \\ \{\nu_t\}_{t \in \operatorname{Im}\nu} &= \{r_1, \mathbb{R}^2\}. \end{aligned}$$

Therefore, μ and ν are similar, but they are not similar to η.
3. *Considering Proposition 5, it follows that $\mu \simeq \eta \simeq \nu$.*

These three equivalence relations identify classes where fuzzy subspaces share an affinity according to the given criteria. It should be noted that if any of the above three relations relate to μ and η, their level sets are within the same variety of flags of a given type, i.e.,

$$\{\mu_t\}_{t \in \operatorname{Im}\mu}, \{\eta_s\}_{s \in \operatorname{Im}\eta} \in \mathcal{L}_q((d_1, \ldots, d_m), n),$$

for some dimensions $\{d_i\} = \{\dim(\mu_t) : t \in \operatorname{Im}\mu\} = \{\dim(\eta_s) : s \in \operatorname{Im}\eta\}$.

5. Classification and Enumeration of Equivalence Classes on Finite-Dimensional Vector Spaces

Note that the flags are the building blocks in flag codes, being their generalisation of the fuzzy vector subspaces. As discussed earlier, the previous three equivalence relations provide a way to define classes of fuzzy vector subspaces, where members of such classes exhibit identical characteristics. This perspective is particularly important in the context of flag codes because flags belonging to the same class can be considered practically identical and interchangeable.

In the field of coding, an important issue is to determine the actual number of different available codewords that can be used in the transmission of information. In the flag code strategy, the codewords are flags. Hence, the interest is in finding out the amount of distinct flags in a given vector space. In this section, this issue is treated from the general perspective of fuzzy linear algebra. From this point, determining the number of essentially distinct flags is equivalent to classifying and enumerating distinct equivalence classes in each of the previously defined relations based on the mentioned criteria. Therefore, this study will investigate the number of equivalence classes formed under each criterion from an algebraic perspective.

In this work, the study will stick to finite-dimensional vector spaces, due to their direct connection to the network coding problem. The fundamental theorem of vector spaces states that any finite-dimensional vector space E over a field $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$, where n is the dimension of the vector space E. Due to this, throughout the paper, every vector space will be treated as $E = \mathbb{F}^n$.

Proposition 6. *There are infinite equivalence classes of isomorphic fuzzy vector subspaces of $\mathbb{F}^n$ for any $n \in \mathbb{N}$. Furthermore,*

$$\left|\mathcal{F}(\mathbb{F}^n)/\cong\right| \geq |\mathbb{R}|.$$

Proof. It suffices to prove the second part of the statement. Let us consider the mapping

$$\begin{array}{rccc} M: & [0,1] & \longrightarrow & \mathcal{F}(\mathbb{F}^n)/\cong \\ & r & \mapsto & [\mu]_{\cong} \end{array}$$

where μ is the fuzzy set $\mu(\mathbb{F}^n) = r$, which is trivially a fuzzy vector subspace. Thus, M maps each value in $[0,1]$ to the equivalence class of the constant fuzzy set μ defined above. It suffices to prove that this mapping is one-to-one. Assume $r, s \in [0,1]$, $r \neq s$ and $M(r) = [\mu]_{\cong}$, $M(s) = [\eta]_{\cong}$, for some $\mu, \eta \in \mathcal{F}(\mathbb{F}^n)$, and let us show that it is $[\mu]_{\cong} \neq [\eta]_{\cong}$. By reductio ad absurdum, suppose that both equivalence classes coincide, which means that there exists an isomorphism $f : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ satisfying Equation (1). Since η is a constant mapping, for each $x \in \mathbb{F}^n$, thus, $\eta(f(x)) = \eta(x)$. Consequently,

$$r = \mu(x) = \eta(f(x)) = \eta(x) = s,$$

which contradicts the hypothesis. Hence, M is one-to-one, and therefore

$$|\mathbb{R}| = |[0,1]| = |M([0,1])| \leq |\mathcal{F}(\mathbb{F}^n)/\cong|.$$

□

Remark 1. *Note that Proposition 6 can be readily extended to infinite-dimensional vector spaces.*

Regarding the similarity relation $\sim$, it can be observed that, for two fuzzy vector subspaces μ, η, if $\mu \sim \eta$, then the (extended) flags of μ and η coincide. In other words, there is a bijection between equivalence classes and (possibly extended) flags. This means that this relation represents a criterion that identifies fuzzy vector subspaces with the same associated flag. The next result presents a lower bound for the number of equivalence classes (number of extended flags), in terms of the cardinality of the field $\mathbb{F}$.

Proposition 7. *Let $\mathbb{F}$ be a field and $n \geq 2$. Then,*

$$\left|\mathcal{F}(\mathbb{F}^n)/\!\sim\right| \geq |\mathbb{F}|.$$

Proof. Let us build a one-to-one mapping from $\mathbb{F}$ to the quotient set, which will imply this result.

Consider, for $r \in \mathbb{F}$, the vector $\boldsymbol{v}_r = (r, 1, 0, \dots, 0)$, expressed in canonical coordinates, and denote by $\langle \boldsymbol{v}_r \rangle$ the subspace of $\mathbb{F}^n$ generated by $\boldsymbol{v}_r$. It is trivial to show that $\langle \boldsymbol{v}_r \rangle \neq \langle \boldsymbol{v}_s \rangle$ whenever $r \neq s$. Now, define the fuzzy set

$$\mu_r(x) = \begin{cases} 0.5 & \text{if} \quad x \in \langle \boldsymbol{v}_r \rangle, \\ 0 & \text{if} \quad x \notin \langle \boldsymbol{v}_r \rangle. \end{cases}$$

Clearly, the set of level sets of μ_r is $\{\langle \boldsymbol{v}_r \rangle, \mathbb{F}^n\}$. Based on Proposition 1, μ_r is a well-defined fuzzy vector subspace.

Let us define the mapping $M : \mathbb{F} \longrightarrow \mathcal{F}(\mathbb{F}^n)/\sim$ as $M(r) = [\mu_r]_\sim$. Clearly, M is one-to-one since, given $r \neq s$, the level sets of μ_r and μ_s are not equal since $\langle \boldsymbol{v}_r \rangle \neq \langle \boldsymbol{v}_s \rangle$; therefore, $M(r) = [\mu_r]_\sim \neq [\mu_s]_\sim = M(s)$. □

Corollary 2. *If $|\mathbb{F}| = \infty$, then the number of equivalence classes for the similarity relation is also infinite.*

Corollary 3. *There are infinite equivalence classes of similar fuzzy vector subspaces of $\mathbb{F}^n$ for $n \geq 2$, where $\mathbb{F}$ is a field with characteristic 0.*

Proof. This is an immediate consequence of every field of characteristic 0 being infinite and applying the previous result. □

For the sake of completeness, this study has been conducted in general, encompassing the case of infinite fields, although the interest is in finite fields, which are used in flag codes. In addition, note that Theorem 2 will provide an exact value for the cardinality of the quotient set induced by the similarity relation, in the case of a field of cardinality $|\mathbb{F}| = q$, q being a prime power.

Although the case $n = 1$ is of little practical interest, the next result stating the cardinality of the corresponding quotient set is provided here for the sake of completeness.

Proposition 8. *There are two equivalence classes of similar fuzzy vector subspaces of $\mathbb{F}$, that is, $|\mathcal{F}(\mathbb{F})/\!\sim| = 2$.*

Proof. Taking into account Proposition 1, the level sets of a fuzzy vector subspace μ of $\mathbb{F}$ must be vector subspaces of $\mathbb{F}$. Since $\mathbb{F}$ is a field, its only vector subspaces play the role of ideals, and being a field, the only ideals are $\{\mathbf{0}\}$ and $\mathbb{F}$. This implies that the level set μ_t is either $\mathbb{F}$ or $\{\mathbf{0}\}$ for each $t \in [0, 1]$. Therefore, there are only two possible (extended) flags, $\{\mathbb{F}\}$ and $\{\{\mathbf{0}\} \subsetneq \mathbb{F}\}$. Since an equivalence class for the similarity relation is completely characterised by its extended flag, there are only two equivalence classes. □

Remark 2. *In Proposition 8, it is noteworthy that the field can be either finite or infinite, in contrast to Corollary 2, which is exclusively applicable to infinite fields. Towards the conclusion of the paper, this result is extended to n-dimensional vector spaces over finite fields (refer to Theorem 2). Additionally, Table 1 illustrates this particular case for $n = 1$.*

Theorem 1. *Let $\mathbb{F}$ be a field. Then, there are 2^n equivalence classes of iso-similar fuzzy vector subspaces of $\mathbb{F}^n$.*

Proof. For each fuzzy vector subspace $\mu : \mathbb{F}^n \longrightarrow [0,1]$, let us consider its chain of level sets. Since the dimension of $\mathbb{F}^n$ is n, hence there are at most $n+1$ crisp subspaces in a chain. Then, for some values $t_i \in [0,1]$, the level sets of μ are

$$\mu_{t_1} \subseteq \mu_{t_2} \subseteq \mu_{t_3} \subseteq \cdots \subseteq \mu_{t_{k-1}} \subseteq \mu_{t_k},$$

where $k \leq n+1$ and $t_i > t_{i+1}$. Let us consider the set

$$\mathcal{S}_n := \{x \subseteq \{0,1,\ldots,n\} : n \in x\}$$

and consider the mapping $h : \mathcal{F}(\mathbb{F}^n) \longrightarrow \mathcal{S}_n$, defined by

$$h(\mu) = \{\dim(\mu_t) \mid t \in [0, \mu(\mathbf{0})]\}.$$

The function h is well defined because the level set μ_t is not empty when $t \in [0, \mu(\mathbf{0})]$ and for $t = 0$,

$$\dim(\mu_0) = \dim(\mathbb{F}^n) = n.$$

Moreover, the following properties hold:

1. The function h is surjective. Given $O = \{d_1, d_2, \ldots, d_m\} \in \mathcal{S}_n$, any (extended) flag of *signature* $(d_1, d_2, \ldots, d_m)$ can be considered

$$U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots \subseteq U_m = \mathbb{F}^n,$$

so that each $d \in O$ is reached. Then, choosing m values of the unit interval $t_1 > t_2 > \cdots > t_m$, a fuzzy vector subspace can be considered defined as

$$\mu(x) = \begin{cases} t_1 & \text{if} \quad x \in U_1, \\ t_i & \text{if} \quad x \in U_i \setminus U_{i-1}. \end{cases}$$

By construction, $h(\mu) = O$.

2. If μ is iso-similar to η, then $h(\mu) = h(\eta)$. By hypothesis, there exists an isomorphism $f : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ satisfying that μ and $f(\eta)$ have the same level sets. Since the dimensions of the level sets of $f(\eta)$ and η are equal because f is an isomorphism, therefore the level sets of μ and η have the same dimensions, that is, $h(\mu) = h(\eta)$.

3. If μ is not iso-similar to η, then $h(\mu) \neq h(\eta)$. By contradiction, suppose that $h(\mu) = h(\eta)$, that is,

$$\{\dim(\mu_t) \mid t \in [0, \mu(\mathbf{0})]\} = \{\dim(\eta_s) \mid s \in [0, \eta(\mathbf{0})]\}.$$

This implies that they have the same number of level sets and their dimensions are equal. Due to this, for each crisp subspace μ_{t_i}, there is an isomorphism f_i such that $f_i(\mu_{t_i}) = \eta_{s_i}$, where η_{s_i} is the crisp level set of η with the same dimension of μ_{t_i}. In particular, f_1 is an isomorphism between μ_{t_1} and η_{s_1}. Extending the basis from μ_{t_1} and η_{s_1} to μ_{t_2} and η_{s_2}, respectively, another isomorphism f_2 can be built between μ_{t_2} and η_{s_2} satisfying that $f_2(\mu_1) = f_1(\mu_1)$. This procedure can be repeated a finite number of times to reach the greatest level set of them, building an isomorphism f_k between μ_{t_k} and η_{s_k} satisfying $f_k(\mu_{t_i}) = f_i(\mu_{t_i})$ for all $i < k$. Since the greatest level sets of μ and η are $\mu_0 = \mathbb{F}^n = \eta_0$, an isomorphism $f_k : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ such that

$$\{\mu_t \mid t \in \operatorname{Im}\mu\} = \{f_k(\eta_s) \mid s \in \operatorname{Im}\eta\},$$

has been found, which is a contradiction because $\mu \not\simeq \eta$.

Taking into account the previous properties of h, the following mapping can be defined

$$\hat{h} : \mathcal{F}(\mathbb{F}^n)/\simeq \longrightarrow \mathcal{S}_n,$$

defined by $\hat{h}([\mu]) = h(\mu)$, where $[\mu]$ is the class of μ. It is well defined because if $\mu \simeq \eta$, then $h(\mu) = h(\eta)$. Moreover, $\hat{h}$ is one-to-one because the images of different equivalence classes are different. Since h is surjective, $\hat{h}$ is surjective. Therefore, there is a bijection between the equivalence classes of iso-similar fuzzy vector spaces and the set $\mathcal{S}_n$, which has 2^n elements. □

At this point, the fuzzy vector subspaces according to the partitions induced by the isomorphism and the iso-similarity relations have been enumerated for vector spaces on both finite and infinite fields, but for the similarity relation, the results have been proved considering only vector spaces on infinite fields.

In the following, the study will be focused on the classification of fuzzy vector subspaces in a vector space over a finite field, considering the similarity relation, which requires the utilisation of concepts from various mathematical areas. First, let us recall the notion of Gaussian binomial coefficients [35]:

Proposition 9 ([36]). *Let $\mathbb{F}$ be a finite field with q elements. The number of subspaces of dimension k in $\mathbb{F}^n$ is given by the Gaussian binomial coefficient*

$$\binom{n}{k}_q := \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

Example 3. *In the context of enumerating fuzzy vector spaces on $\mathbb{F}^n$, Proposition 8 addresses the case $n = 1$. Let us make a constructive proof of the fact that there are $2q + 4$ similar fuzzy vector spaces on $\mathbb{F}^2$, where $\mathbb{F}$ is a finite field of cardinality q. This same idea will be used in the main result of this section, Theorem 2.*

It has been established that each chain of vector subspaces finishing in $\mathbb{F}^2$ generates an equivalence class of similar fuzzy vector spaces. Since the dimension of $\mathbb{F}^2$ is 2, the dimension of a crisp subspace can be $0, 1,$ or 2. Trivially, there is only one 0-dimension subspace and one 2-dimension subspace. One can easily check that the 1-dimensional subspaces are generated by one vector that can be chosen from $\{(1,0),(x,1) : x \in \mathbb{F}\}$ (just by scaling any non-zero vector in the corresponding subspace). Thus, there are $1 + q = \binom{2}{1}_q$ 1-dimension subspaces. Considering $r \subseteq \mathbb{F}^2$ any 1-dimension subspace, all the possible flags (chains ending in $\mathbb{F}^2$) of subspaces are as follows:

1. $\{\mathbf{0}\} \subseteq r \subseteq \mathbb{F}^2$*, which has $1 + q$ possibilities for r.*
2. $r \subseteq \mathbb{F}^2$*, which also has $1 + q$ possibilities.*
3. $\{\mathbf{0}\} \subseteq \mathbb{F}^2$*, which is 1 possibility.*
4. $\mathbb{F}^2$*, which is 1 possibility.*

By adding up all of these possibilities, there are $2(q + 1) + 2$ possible chains to build a class of similar fuzzy vector spaces.

Note that, in this constructive proof, the whole set of flags of the vector space $\mathbb{F}^2$ has been built. This strategy will be used in the main result of this section to determine the number of equivalence classes of a finite-dimensional vector space over a finite field. Prior to the statement of that theorem, some technical results and definitions are required.

Lemma 1. *Let $\mathbb{F}$ be a finite field with q elements and consider a k-dimensional subspace V of $\mathbb{F}^n$. The number of distinct subspaces W of dimension $k + p \leq n$ (for $1 \leq p \leq n - k$) satisfying $V \subseteq W$ is*

$$\binom{n-k}{p}_q.$$

Proof. To count the number of possible W satisfying

$$V \subseteq W \subseteq \mathbb{F}^n,$$

where W is a $(k+p)$-dimensional subspace of $\mathbb{F}^n$, the strategy will be taking the quotient by V; therefore,

$$V/V \subseteq W/V \subseteq \mathbb{F}^n/V,$$

equivalently, since $\mathbb{F}^n/V \cong \mathbb{F}^{n-k}$,

$$\{0\} \subseteq W' \subseteq \mathbb{F}^{n-k},$$

where W' is a p-dimensional subspace. Thus, it suffices to count the number of possible W', which is given by the direct application of Proposition 9 since the number of p-dimensional subspaces of $\mathbb{F}^{n-k}$ is $\binom{n-k}{p}_q$. $\square$

Another notion needed is that of ordered partition, which will play an important role in the computation of the number of flags in the vector space.

Definition 7 ([37]). *An* ordered partition *or a* composition *of a number $n \in \mathbb{N}$ is a tuple of natural numbers adding up to n. The set of all ordered partitions of n is denoted by*

$$\mathcal{P}_n := \left\{(k_1, k_2, \dots, k_m) \mid \sum k_i = n, 1 \leq k_i \leq n\right\}.$$

Notation 2. *For ease of presentation, the elements of $\mathcal{P}_n$ are denoted by underlined lower-case letters as in $\underline{k} := (k_1, k_2, \dots, k_m)$.*

The relation between ordered partitions and flags of subspaces is clear when one considers, for $\underline{k} = (k_1, k_2, \dots, k_m) \in \mathcal{P}_n$, the set

$$C(\underline{k}) := \left\{U_1 \subsetneq U_2 \subsetneq \dots \subsetneq U_m \mid U_i \text{ subspace of } \mathbb{F}^n, \dim(U_i) = \sum_{j \leq i} k_j, \text{ for all } i\right\},$$

which consists of all the flags with a given *configuration* of dimensions since, given an element $\underline{k} \in \mathcal{P}_n$, one can build multiple flags where the dimensions of the subspaces are induced by $\underline{k}$:

1. The number of vector subspaces is determined by the length of the tuple $\underline{k}$, that is, $|\underline{k}| = m$.
2. The dimension of U_i $(1 \leq i \leq n)$ is equal to the sum of the first i elements of $\underline{k}$.

In particular, it is $U_m = \mathbb{F}^n$ for all $\underline{k} = (k_1, \dots, k_m) \in \mathcal{P}_n$. Hence,

$$C(\underline{k}) = \mathcal{L}_q((k_1, k_1 + k_2, \dots, \sum_{i<m} k_i), n).$$

In the rest of this work, ordered partitions in the technical proofs will be used as a schematic representation of the variety of flags, as can be seen in the following example.

Example 4. *Let us consider the set of ordered partitions of* 4*:*

$$\mathcal{P}_4 = \{(1,1,1,1), (2,1,1), (1,2,1), (1,1,2), (2,2), (3,1), (1,3), (4)\},$$

and take $\underline{k} = (2,1,1)$ and $\underline{k'} = (1,3)$. These two ordered partitions induce the following configurations of dimensions: for $\underline{k}$, $d_1 = 2 < d_2 = 2+1 = 3 < d_3 = 2+1+1 = 4$, and for $\underline{k'}$, $d'_1 = 1 < d'_2 = 1+3 = 4$. Thus, in this case,

$$C(\underline{k}) = \{U_1 \subsetneq U_2 \subsetneq \mathbb{F}^4 \mid U_1, U_2 \textit{ subspace of } \mathbb{F}^4, \dim(U_1) = 2, \dim(U_2) = 3\} = \\ = \mathcal{L}_q((2,3),4),$$

and

$$C(\underline{k'}) = \{U_1 \subsetneq \mathbb{F}^4 \mid U_1 \textit{ subspace of } \mathbb{F}^4, \dim(U_1) = 1\} = \\ = \mathcal{L}_q((1), 4).$$

The question that arises naturally is about the cardinality of $C(\underline{k})$ since it will be of great help in the determination of the number of equivalence classes for the similarity relation. The answer to this question is included in the following lemma.

Lemma 2. *Let us consider $\mathbb{F}^n$, where $\mathbb{F}$ is a field of cardinality q. Then, for every ordered partition $\underline{k} = (k_1, \dots, k_m) \in \mathcal{P}_n$,*

$$|C(\underline{k})| = \binom{n}{k_1}_q \cdot \binom{n-k_1}{k_2}_q \cdot \binom{n-k_1-k_2}{k_3}_q \cdot \dots \cdot \binom{n-\sum_{j=1}^{m-1} k_j}{k_m}_q.$$

Proof. Let us take $\underline{k} = (k_1, k_2, \dots, k_m) \in \mathcal{P}_n$, and consider the possible flags in $C(\underline{k})$, which are of the form

$$U_1 \subsetneq U_2 \subsetneq U_3 \cdots \subsetneq U_m = \mathbb{F}^n,$$

where $\dim(U_i) = \sum_{j=1}^{i} k_j$. The number of possible subspaces that can be in the first position, that is, that can act as U_1, are those of dimension k_1. Thus, $\binom{n}{k_1}_q$ possibilities for U_1. For any $i > 1$, based on Lemma 1, the number of possible subspaces for U_i is

$$\binom{n-\dim(U_{i-1})}{\dim(U_i)-\dim(U_{i-1})}_q = \binom{n-\sum_{j<i} k_j}{k_i}_q.$$

Hence, the aggregate number of possibilities for such chains of vector subspaces (and therefore the cardinality of $C(\underline{k})$) is

$$\binom{n}{k_1}_q \cdot \binom{n-k_1}{k_2}_q \cdot \binom{n-k_1-k_2}{k_3}_q \cdot \dots \cdot \binom{n-\sum_{j=1}^{m-1} k_j}{k_m}_q.$$

□

Notation 3. *Following the classical notation [38], in the rest of the paper, the following notation will be used,*

$$\binom{n}{\underline{k}}_q := \binom{n}{k_1}_q \cdot \binom{n-k_1}{k_2}_q \cdot \binom{n-k_1-k_2}{k_3}_q \cdot \dots \cdot \binom{n-\sum_{j=1}^{m-1} k_j}{k_m}_q.$$

These results allow us to state the following:

Theorem 2. *Let $\mathbb{F}$ be a finite field with q elements. Then, the number of equivalence classes for the similarity relation on $\mathbb{F}^n$, which will be denoted by $\Phi_n(q)$, is given by*

$$\Phi_n(q) := 2 \sum_{\underline{k} \in \mathcal{P}_n} \binom{n}{\underline{k}}_q \tag{5}$$

Proof. For the similarity relation, each equivalence class is completely characterised by a flag of vector subspaces finishing in $\mathbb{F}^n$, so it suffices to count the number of such flags. Notice that, given $\underline{k} \in \mathcal{P}_n$, every flag considered in $C(\underline{k})$ starts with a non-zero subspace. Since all extended flags need to be considered too, starting in $\{\mathbf{0}\}$, let us define

$$C_0(\underline{k}) := \big\{ \{\mathbf{0}\} \subsetneq U_1 \subsetneq U_2 \subsetneq \dots \subsetneq \mathbb{F}^n \mid U_1 \subsetneq U_2 \subsetneq \dots \subsetneq \mathbb{F}^n \in C(\underline{k}) \big\}.$$

It is clear to see that $|C_0(\underline{k})| = |C(\underline{k})|$ and that $C(\underline{k_1}) \neq C(\underline{k_2})$ if $\underline{k_1} \neq \underline{k_2}$.

Thus, the set of all flags in $\mathbb{F}^n$ can be written as $\mathcal{C} := \bigcup_{\underline{k}\in\mathcal{P}_n} C(\underline{k}) \cup C_0(\underline{k})$, with all unions being disjointed. For the reasons stated above, it follows that

$$\Phi_n(q) = |\mathcal{C}| = \sum_{\underline{k}\in\mathcal{P}_n} (|C(\underline{k})| + |C_0(\underline{k})|) = 2\sum_{\underline{k}\in\mathcal{P}_n} \binom{n}{\underline{k}}_q,$$

as a result of applying Lemma 2. □

For the sake of completeness, let us show the value of $\Phi_n(q)$ for $1 \leq n \leq 6$:

$$\Phi_1(q) = 2$$
$$\Phi_2(q) = 2(q+2)$$
$$\Phi_3(q) = 2\left(q^3 + 4q^2 + 4q + 4\right)$$
$$\Phi_4(q) = 2\left(q^6 + 6q^5 + 12q^4 + 18q^3 + 18q^2 + 12q + 8\right)$$
$$\Phi_5(q) = 2\left(q^{10} + 8q^9 + 24q^8 + 48q^7 + 76q^6 + 92q^5 + 100q^4 + 84q^3 + 60q^2 + 32q + 16\right)$$
$$\Phi_6(q) = 2\left(q^{15} + 10q^{14} + 40q^{13} + 102q^{12} + 206q^{11} + 334q^{10} + 478q^9 + 596q^8 + 658q^7 + \right.$$
$$\left. + 650q^6 + 572q^5 + 448q^4 + 300q^3 + 176q^2 + 80q + 32\right)$$

The results of computing $\Phi_n(q)$ for specific values of q and n are shown in Table 1.

Table 1. Values of $\Phi_q(n)$.

	n					
q	**1**	**2**	**3**	**4**	**5**	**6**
2	2	8	72	1392	55,616	4,515,776
3	2	10	158	7702	1,135,466	503,580,226
4	2	12	296	29,616	11,885,216	19,093,016,256
5	2	14	498	89,286	80,130,602	359,667,188,414
7	2	18	1142	508,902	1,587,898,106	34,684,388,002,986
8	2	20	1608	1,036,752	5,348,388,896	220,736,678,071,616
9	2	22	2186	1,958,326	15,790,793,306	1,145,969,918,373,286
11	2	26	3726	5,879,670	102,064,084,682	19,488,961,838,915,664
13	2	30	5858	14,880,102	491,376,001,226	210,943,790,745,481,792

The examples above show how the number of classes of similar fuzzy vector subspaces grows rapidly with n. From a practical point of view, the growth rate of the polynomial might be discussed. Nevertheless, optimality and computation-based results are more focused on finding the right flag than on searching all the flags [8]. Thus, the focus is now on the algebraic structure of the quotient sets, which would ease running over the classes to search for a particular one. This topic is studied in the next section.

6. Absence of Algebraic Structure of the Quotient Sets Induced by the Equivalence Relations

In the crisp case, it is easy to see that, under an equivalence relation (the isomorphism relation), the quotient set preserves the structure of the lattice, provided the infimum is given by the intersection and the supremum is the sum. Our interest now is in studying whether this property extends to fuzzy vector subspaces equipped with the three equivalence relations defined in Section 4.

It has been proved in the literature that the study of flags provides interesting applications, e.g., information theory [5] or network coding [6]. Thus, the identification of fuzzy vector spaces with flags [9] links the study of fuzzy vector spaces with these topics. As a matter of fact, in flag codes Section 2, the set of flags is a meet-semilattice with the intersection. Nevertheless, from the algebraic point of view, equivalence relations are defined to take a quotient that keeps the structure in some manner. This is the topic covered in this section, where it will be shown that none of the standard equivalence relations considered in the framework of fuzzy subgroups extend properly to fuzzy vector spaces. In particular, the examples provided will illustrate that the intersection and sum of fuzzy vector subspaces are not well-defined operations on the set of equivalence classes.

The idea is to check that the $\wedge$ and $+$ operations cannot be defined properly on the quotient sets. This means that the operations do not depend on the representative chosen in each equivalence class. In Section 4, three equivalence relations have been introduced, and this section is split into three parts, one for each analysis.

6.1. The Isomorphism Relation

Even though this relation resembles the idea of isomorphism in fuzzy vector spaces, the next example shows that it does not behave well with the quotient set, that is, the set of equivalence classes by the isomorphism relation does not form a lattice with the subsethood relation. In particular, the example shows that the intersection and sum of fuzzy vector subspaces in the quotient set depend on the representative chosen in each class.

Example 5. *Let us consider the real vector space $\mathbb{R}^2$ and define the fuzzy sets $\mu, \eta : \mathbb{R}^2 \longrightarrow [0,1]$ as follows:*

$$\mu(x,y) = \begin{cases} 0.5 & \text{if} \quad x = 0, \\ 0.2 & \quad\quad \text{otherwise,} \end{cases}$$

$$\eta(x,y) = \begin{cases} 0.5 & \text{if} \quad y = 0, \\ 0.2 & \quad\quad \text{otherwise,} \end{cases}$$

They are isomorphic fuzzy vector subspaces considering the isomorphism $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined as $f(x,y) = (y,x)$. Moreover, the minimum of them is

$$(\mu \wedge \eta)(x,y) = \mu(x,y) \wedge \eta(x,y) = \begin{cases} 0.5 & \text{if} \quad x = y = 0, \\ 0.2 & \quad\quad \text{otherwise,} \end{cases}$$

Trivially, $\mu \wedge \eta$ cannot be isomorphic to either μ or η because the only element that, under $\mu \wedge \eta$, maps to 0.5 is $(0,0)$.

Therefore, it has been shown that $[\mu]_{\cong} = [\eta]_{\cong}$ does not imply $[\mu \wedge \eta]_{\cong} = [\mu \wedge \mu]_{\cong}$, so the operation $[\mu]_{\cong} \wedge [\eta]_{\cong} = [\mu \wedge \eta]_{\cong}$ is not well defined on the quotient set $\mathbb{R}^2/\cong$. In addition, the sum is not well defined either as shown below.

The sum $\mu + \eta$ is

$$(\mu + \eta)(x,y) := \sup\{\mu(x_1,y_1) \wedge \eta(x_2,y_2) \mid (x,y) = (x_1,y_1) + (x_2,y_2)\},$$

equivalently,

$$(\mu + \eta)(x,y) := \sup\{\mu(x',y') \wedge \eta(x - x', y - y') \mid x', y' \in \mathbb{R}\}.$$

On the one hand, for any $(a,b) \in \mathbb{R}^2$,

$$(\mu + \eta)(a,b) \geq \mu(0,b) \wedge \eta(a,0) = 0.5 \wedge 0.5 = 0.5.$$

On the other hand, $(\mu + \eta)(a,b) \leq 0.5$ for any $(a,b) \in \mathbb{R}^2$. Therefore,

$$(\mu + \eta)(x,y) = 0.5$$

for all $(x,y) \in \mathbb{R}^2$. Consequently, $\mu + \eta$ is not isomorphic to either μ or η. In other terms, $[\mu]_{\cong} + [\eta]_{\cong} \neq [\mu]_{\cong} + [\mu]_{\cong}$ and again $[\mu]_{\cong} + [\eta]_{\cong} = [\mu + \eta]_{\cong}$ is not a well-defined operation.

6.2. The Similarity Relation

Now, two examples are presented which show that both $\wedge$ and $+$ are not well defined in the quotient set induced by the similarity relation $\sim$.

Example 6. *Let us consider $\mathbb{F}^n = \mathbb{R}^2$ and, for all, $(x, y) \in \mathbb{R}^2$,*

$$\mu_1(x, y) = \begin{cases} 1 & \textit{if } y = 0 \\ 0.9 & \textit{otherwise}, \end{cases}$$

$$\mu_2(x, y) = \begin{cases} 0.1 & \textit{if } y = 0 \\ 0 & \textit{otherwise}, \end{cases}$$

and

$$\eta(x, y) = 0.5.$$

It is easy to see that $\mu_1 \sim \mu_2$ since, for both fuzzy vector subspaces, their common set of t-cuts is $\{OX, \mathbb{R}^2\}$, where $OX = \{(x, 0) : x \in \mathbb{R}\}$. Let us compute the infima $\mu_1 \wedge \eta$ and $\mu_2 \wedge \eta$: let $(x, y) \in \mathbb{R}^2$, then

$$(\mu_1 \wedge \eta)(x, y) = 0.5 = \eta(x, y),$$

and

$$(\mu_2 \wedge \eta)(x, y) = \begin{cases} 0.1 & \textit{if } y = 0 \\ 0 & \textit{otherwise} \end{cases} = \mu_2(x, y).$$

Since $\eta \not\sim \mu_2$, because the set of t-cuts of η is simply $\{\mathbb{R}^2\}$, which yields $\mu_1 \wedge \eta \not\sim \mu_2 \wedge \eta$, although $\mu_1 \sim \mu_2$. That is, the operation $\wedge$ depends on the representative element selected from the equivalence class; therefore, this operation is not well defined in the quotient set $\mathbb{R}^2/\sim$.

Similarly, an analogous study is now carried out for the sum operation:

Example 7. *Let us consider the real vector space $\mathbb{R}^2$ and define the fuzzy sets $\mu, \eta : \mathbb{R}^2 \longrightarrow [0, 1]$ as follows:*

$$\mu(x, y) = \begin{cases} 0.8 & \text{if} \quad x = 0, \\ 0.6 & \quad\quad \text{otherwise}, \end{cases}$$

$$\eta(x, y) = \begin{cases} 0.6 & \text{if} \quad x = 0, \\ 0.4 & \quad\quad \text{otherwise}, \end{cases}$$

Their level sets are equal, so μ is similar to η. Moreover, the sum $\mu + \eta$ is

$$(\mu + \eta)(x, y) := \sup\{\mu(x_1, y_1) \wedge \eta(x_2, y_2) \mid (x, y) = (x_1, y_1) + (x_2, y_2)\},$$

equivalently,

$$(\mu + \eta)(x, y) := \sup\{\mu(x', y') \wedge \eta(x - x', y - y') \mid x', y' \in \mathbb{R}\}.$$

On the one hand, since $\mu(x, y) \leq 0.8$ and $\eta(x, y) \leq 0.6$, the sum is upper bounded by

$$(\mu + \eta)(x, y) \leq 0.8 \wedge 0.6 = 0.6,$$

for any pair $(x, y) \in \mathbb{R}^2$. On the other hand, fixed $(x, y) \in \mathbb{R}^2$,

$$\sup\{\mu(x', y') \wedge \eta(x - x', y - y') \mid x', y' \in \mathbb{R}\} \geq \mu(x, 0) \wedge \eta(0, y) \geq 0.6 \wedge 0.6 = 0.6.$$

Consequently, $\mu + \eta(x, y) = 0.6$ for all $(x, y) \in \mathbb{R}^2$ and it follows that the sum is not similar to either μ or η. Again, as in the previous example, it is $\mu \sim \eta$ but $\mu + \mu \not\sim \mu + \eta$, so the operation $+$ is not well defined on the quotient set $\mathbb{R}^2/\sim$.

As a conclusion, in general, $\wedge$ and $+$ cannot be extended from the lattice of fuzzy vector subspaces to the quotient set induced by the similarity relation.

6.3. The Iso-Similarity Relation

As proved in Proposition 5, every pair of fuzzy vector spaces related by the isomorphism or the similarity relations are also related by the iso-similarity relation. This means that the examples in the previous two subsections are also counterexamples which show that $\wedge$ and $+$ cannot be extended to the quotient set.

7. Conclusions and Future Works

In this paper, three equivalence relations defined on the set of fuzzy vector subspaces have been studied, namely, isomorphism, similarity, and iso-similarity. These relations come from fuzzy subgroups and extend the well-established notion of isomorphism to this framework. From an applied point of view, the characterisation of fuzzy vector subspaces in terms of flags of subspaces gives a direct link between these structures and information theory and flag codes. Distinct fuzzy vector subspaces can define the same flag code. From this, it makes sense to study equivalence relations to map the equivalence classes of fuzzy vector spaces to flag codes in a bijective manner. In this line, how many distinct equivalence classes can be defined concerning each equivalence class has been studied.

In the case of the isomorphism or the iso-similarity relation, there is always a fixed amount of equivalence classes independently of the size of the field. The paper continued studying the equivalence classes of the similarity relation. In this case, there are always more equivalence classes than the cardinal of the field. Therefore, whenever the field is infinite, such as a field with characteristic 0, the amount of equivalence classes is also infinite. This result gives an unmanageable number of equivalence classes in practice. However, for finite fields with cardinal q, the exact number of equivalence classes has been computed. The previous analysis shows that the link between information theory and flag codes must be carried out with the similarity relation. This covers the connection between fuzzy vector subspaces and these applications.

On the other hand, from the algebraic point of view, it is interesting to define equivalence relations such that the quotient maintains the structure. As shown in Section 6, the intersection and sum defined by Zadeh's extension principle do not satisfy this. Therefore, as a prospect of future work, a new pair of operations that extend the intersection and sum in order to maintain the lattice structure in the set of equivalence classes is needed, or a new natural equivalence relation that allows the characteristics of the crisp case to be mimicked.

In addition, richer structures might be studied, such as fuzzy closure systems [39], fuzzy graphs and signatures [40,41], or fuzzy rings [42], as well as their connection to network coding, to obtain new, richer strategies based on fuzzy algebraic structures. Additionally, in the context of understanding the algebraic structures, a fruitful research line is the analysis of the aggregation of these structures. Some papers have recently been published on the aggregation of fuzzy algebraic structures considering some of these partitions of fuzzy sets [12,15,43,44]. This current area is being explored and there are many issues with no answer at the moment.

From a more practical point of view, the study presented in this paper deals with the information representativeness that can be achieved by the use of each equivalence relation, that is, in coding terminology, the number of distinct codewords available. As a future work, the aim is to study the optimality of flag codes [8] in terms of properties of the representative elements of the equivalence classes. The expected result will be to define a strategy for building flag codes showing a balance between information representativeness (often requiring a large amount of codewords) and capacity for error prevention and correction (where codewords must be as far away as possible from each other). Another strategy, which is more aligned with the use of fuzzy logic techniques, will focus on creating *fuzzy flag codes*, based on fuzzy vector subspaces, where uncertainty management could allow the generation of more efficient codes for use in network coding.

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